Dr. Häger

MAX-PLANCK-INSTITUT FÜR QUANTENOPTIK

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azimuthal magnetic field

R.F. Schmalz
Max-Planck-Institut für Quantenoptik
D-8046 Garching, Federal Republic of Germany

M.A. Liberman, A.L. Velikovich
Institute for Physical Problems, Moscow, USSR

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On two-dimensional self-similar motion with azimuthal magnetic field

R.F. Schmalz
Max-Planck-Institut für Quantenoptik
D-8046 Garching, Fed.Rep. of Germany

M.A. Liberman, A.L. Velikovich
Institute for Physical Problems,
Moscow, USSR

Abstract

The possibility of two-dimensional self-similar solutions in ideal magneto-hydrodynamics is investigated. A special analytic solution is found which is characterized by spherical expansion of an azimuthally magnetized plasma with adiabatic index 4/3.

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Solutions depending on self-similar coordinates play an essential role in gas dynamics and have important applications in space and laboratory plasma physics. Up to now, mainly highly symmetric models with effectively one space dimension were considered[1]. The question arises, therefore, whether there are simple or even analytically tractable solutions of this type in really two-dimensional situations. In this letter, we answer this question for a class of problems in ideal magnetohydrodynamics with purely azimuthal magnetic field. We pose the general problem and discuss a special case which admits a simple analytical solution. It is characterized by spherical expansion of a magnetized plasma with an adiabatic index of 4/3. The overall symmetry is quite low. Possible applications and the motivation for this work come from situations with a small plasma source and an axis of symmetry as found in plasma generation by a focused laser beam. The magnetic field may be internally generated by various mechanisms discussed in the literature[2].

The underlying model equations are those of ideal magnetohydrodynamics of a hot, dense plasma with an adiabatic index γ:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{v}) = 0 \tag{1}$$

$$\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T + (\gamma-1)\,T\,\nabla\cdot\underline{v} = 0 \tag{2}$$

$$\frac{\partial \underline{b}}{\partial t} - \nabla\times(\underline{v}\times\underline{b}) = 0 \tag{3}$$

$$n\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) + \nabla(nT) + \frac{1}{\beta}\,\underline{b}\times(\nabla\times\underline{b}) = 0 \tag{4}$$

We use normalized quantities with $\beta = \mu_o p_o / b_o^2$ measuring the relative strength of the magnetic field. All variables have the usual meaning. First, we ask in general for the possibility of a separation ansatz of the form $n = N(\underline{\xi})/a^k$, $\underline{b} = \underline{B}(\underline{\xi})/a^\ell$, $T = \Theta(\underline{\xi})/a^m$, and $\underline{v} = \dot{a}\underline{V}(\underline{\xi})$ with the length scale $a = a(t)$ a function of time only and a self-similar coordinate $\underline{\xi} = \underline{r}/a$. Inserting this into eq. (4), we find the conditions

a) $2\ell = k+m$ and
b) $\dot{a}^2 \sim a^{-m}$.

The latter may be solved by a power law $a \sim t^{2/(m+2)}$. Eqs. (1) - (3) give no further restriction on k, ℓ, and m; thus, two of them are free.

For example, k = 3 could be chosen in order to conserve the total mass, or equal to any other value according to the existence of mass sources or sinks depending on the problem.

We now introduce spherical coordinates (r,ϑ,ϕ) and specialize to axial symmetry $\partial/\partial\phi = 0$. We only admit an azimuthal magnetic field $B(\vartheta,\xi)$ with $\xi = r/a$. The velocity components are called $V(\vartheta,\xi)$ and $U(\vartheta,\xi)$ in radial and ϑ direction, respectively. $N(\vartheta,\xi)$ and $\Theta(\vartheta,\xi)$ represent density and temperature. The equations (1) - (4) then transform to

$$kN + \xi \frac{\partial N}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 NV\right) - \frac{1}{\xi \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(NU \sin\vartheta\right) = 0 \qquad (5)$$

$$m\Theta + (\xi - V)\frac{\partial \Theta}{\partial \xi} - U\frac{\partial \Theta}{\partial \vartheta} - (\gamma-1)\Theta\left[\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 V\right) + \frac{1}{\xi \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(U \sin\vartheta\right)\right] = 0 \qquad (6)$$

$$\ell B + \xi \frac{\partial B}{\partial \xi} - \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi BV\right) - \frac{1}{\xi}\frac{\partial}{\partial \vartheta}\left(BU\right) = 0 \qquad (7)$$

$$N\left[\frac{\partial V}{\partial \xi}(V-\xi) + \Gamma V + \frac{U}{\xi}\frac{\partial V}{\partial \vartheta} - \frac{U^2}{\xi}\right] + \frac{\partial}{\partial \xi}(N\Theta) + \frac{1}{\beta}\frac{B}{\xi}\frac{\partial}{\partial \xi}\left(B\xi\right) = 0 \qquad (8)$$

$$N\left[\frac{\partial U}{\partial \xi}(V-\xi) + \Gamma U + \frac{U}{\xi}\frac{\partial U}{\partial \vartheta} + \frac{UV}{\xi}\right] + \frac{1}{\xi}\frac{\partial}{\partial \vartheta}(N\Theta) + \frac{1}{\beta}\frac{B}{\xi \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(B\sin\vartheta\right) = 0 \qquad (9)$$

with $\Gamma = a\ddot{a}/\dot{a}^2 = -m/2$. Eqs. (5)-(9) pose a boundary value problem in two dimensions. The boundary conditions are to be specified depending on the physical situation.

In general, the system (5)-(9) can only be solved numerically. Its overall structure and possible integrals will be studied in a separate paper. However, there are a few simple cases which deserve attention. First, as we are thinking of mostly radial outflow, U could be dropped. Second, V could depend only on ξ. The simplest solution of this type is clearly the homogeneous deformation

$$V = \xi.$$

This solves simultaneously eqs. (5)-(7) if k = 3, ℓ = 2, and γ = 4/3. It is really an exceptional case because we even lose condition b) which is

replaced by $\ddot{a} \sim a^{-2}$. Furthermore, we have the freedom to chose certain functions arbitrarily. Therefore an analytic solution is possible and will be given below.

Other simplifications of the system (5)-(9) are possible. For example, eq. (7) could be made to coincide with eq. (5) by the choice $\ell = k-1$ and $B = N\xi \sin\vartheta$ ("frozen in" fields); for U = 0 the ϑ-dependence could be different. This reduces the number of free parameters to one. We obtain $m = k-2$ and $a \sim t^{2/k}$. Solutions of this type $B \sim N$ are known in planar geometry[3] and apparently can be extended with some modifications to non-planar situations, too.

Another reduction is found by assuming an adiabatic law $\Theta \sim N^{\gamma-1}$ which amounts to having a constant entropy distribution. This is fulfilled in many physical problems and solves eq. (6) for $m = k(\gamma-1)$.

Of special importance is the isothermal case $\gamma = 1$. We only mention here the possibility of a further separation of the angle dependence in this case which leaves us with ordinary differential equations. This will be discussed elsewhere.

We now concentrate on the case $\gamma = 4/3$ and $V = \xi$. Separating the time by $\ddot{a} = \mu a^{-2}$, μ = constant, we obtain (pressure $P = N\Theta$, $\beta = 1$, $\Gamma = -\mu$)

$$\frac{\partial P}{\partial \xi} + \frac{B}{\xi}\frac{\partial}{\partial \xi}\left(\xi B\right) = -\mu \xi N \qquad (10)$$

$$\frac{\partial P}{\partial \vartheta} + \frac{B}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(B\sin\vartheta\right) = 0 \qquad (11)$$

After rescaling, the time evolution law may be solved by

$$t = \pm\left[a^{1/2}(a-s)^{1/2} + s\ln\left(a^{1/2} + (a-s)^{1/2}\right)\right] \qquad (12)$$

with s = sgn μ, positive and negative t corresponding to expansion and compression phases, respectively. For $\mu < 0$, there are also solutions with finite maximal radius

$$t = \pm\left[\arcsin a^{1/2} - a^{1/2}(1-a)^{1/2}\right]$$

and a special power law $a = |t|^{2/3}$. One can show that for $\mu = 0$ there is no nontrivial solution of eqs. (10) and (11). To satisfy eq. (11) we assume for $B(\vartheta,\xi)$ the form

$$B = B_1(\xi)\, f(\vartheta) . \tag{13}$$

Then, the pressure is given by

$$P = P_1(f) + B_1^2 \left[b_0 - \int_0^{\vartheta} \frac{f}{\sin\vartheta'} \frac{d}{d\vartheta'} (f \sin\vartheta')\, d\vartheta' \right] \tag{14}$$

and N may be expressed by B and P via (10). Hence, the solution is given by eqs. (10), (13) and (14) with arbitrary functions $B_1(\xi)$, $P_1(\xi)$, $f(\vartheta)$, and b_0 an arbitrary constant (provided P and N do not change sign). They are determined by the boundary conditions.

To be definite, we give an explicit example with vanishing magnetic field and pressure at a spherical boundary. Here, μ should be positive and the time evolution follows eq. (12) with $s = 1$. For $0 \leq \xi \leq 1$ the solution is given by

$$B = \xi(1-\xi)\sin\vartheta$$

$$N = 6 + (1-\xi)\left[(8 - \sin^2\vartheta)\xi - 4\right]$$

$$P = 3(1-\xi^2) + \xi^2(1-\xi)^2(2 - \sin^2\vartheta) .$$

It is illustrated by Fig. 1 where the lines B = const, N = const, the current lines and P = const. are plotted in sectors 1,2,3 and 4, respectively.

If the magnetic field does not vanish at the plasma boundary, the solution should be supplemented by a corresponding solution of the linear wave equation for B in the vacuum region. The current lines crossing the boundary continue there as displacement currents. The kinetic pressure also may be finite at the boundary if the latter represents a current sheet (tangential discontinuity).

Note that for $\gamma = 4/3$ and homogeneous deformation eq. (9) is satisfied by any $\underline{b} = \underline{B}(\underline{\xi})/a^2(t)$ if $\nabla \cdot \underline{B} = 0$. This allows us to obtain the self-similar solutions for cases of more complex magnetic fields. In particular, a family of such solutions describes expansion (and compression) with constant velocity of arbitrary equilibrium plasma configurations, e.g. solutions of the Grad-Shafranov-Schlüter equations in tokamaks.

References:

1 There are some exceptions, for example
Garlick, A.R., Astrophysics and Space Science 84, 205 (1982).

2 Max, C.E., Manheimer, W.M. and Thomson, J.J., Phys. Fluids 21, 128 (1978).

3 Anderson, D., Bonnedal, M., and Lisak, M., Physica Scripta 22, 507 (1980).

Figure Caption:

An analytic solution for $\gamma = 4/3$. Shown are the lines B = const, N = const, current lines, and P = const in sectors 1,2,3 and 4, respectively.

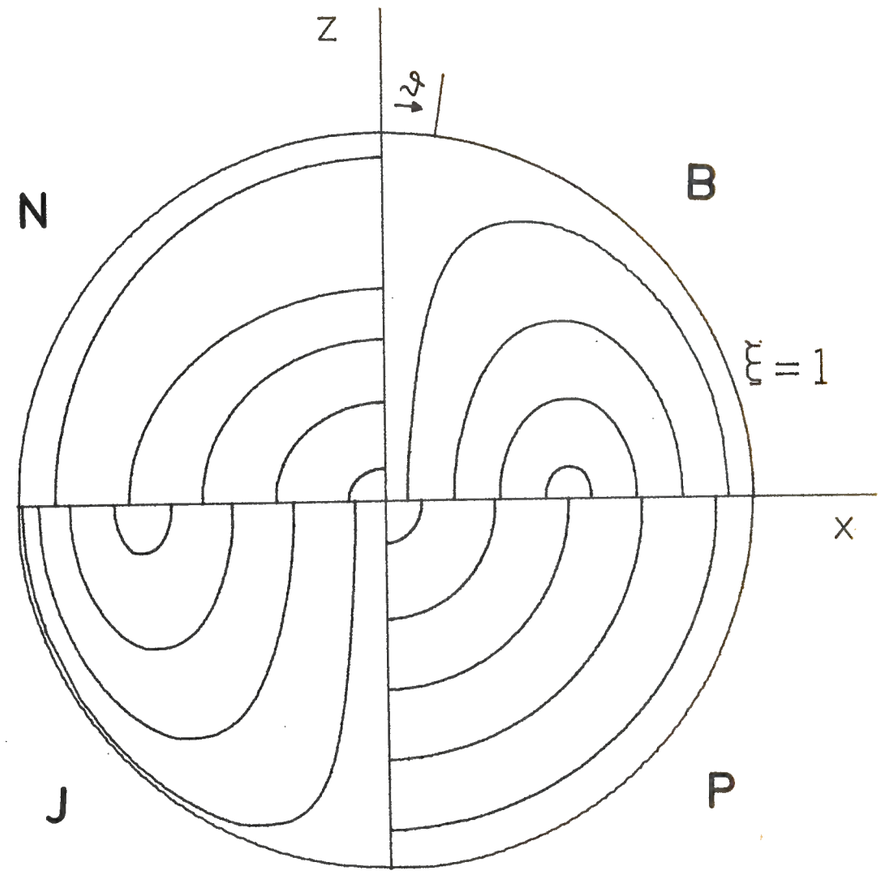

Fig. 1

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