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LINEAR MODE CONVERSION IN LASER PLASMAS

Hans-Jörg Kull

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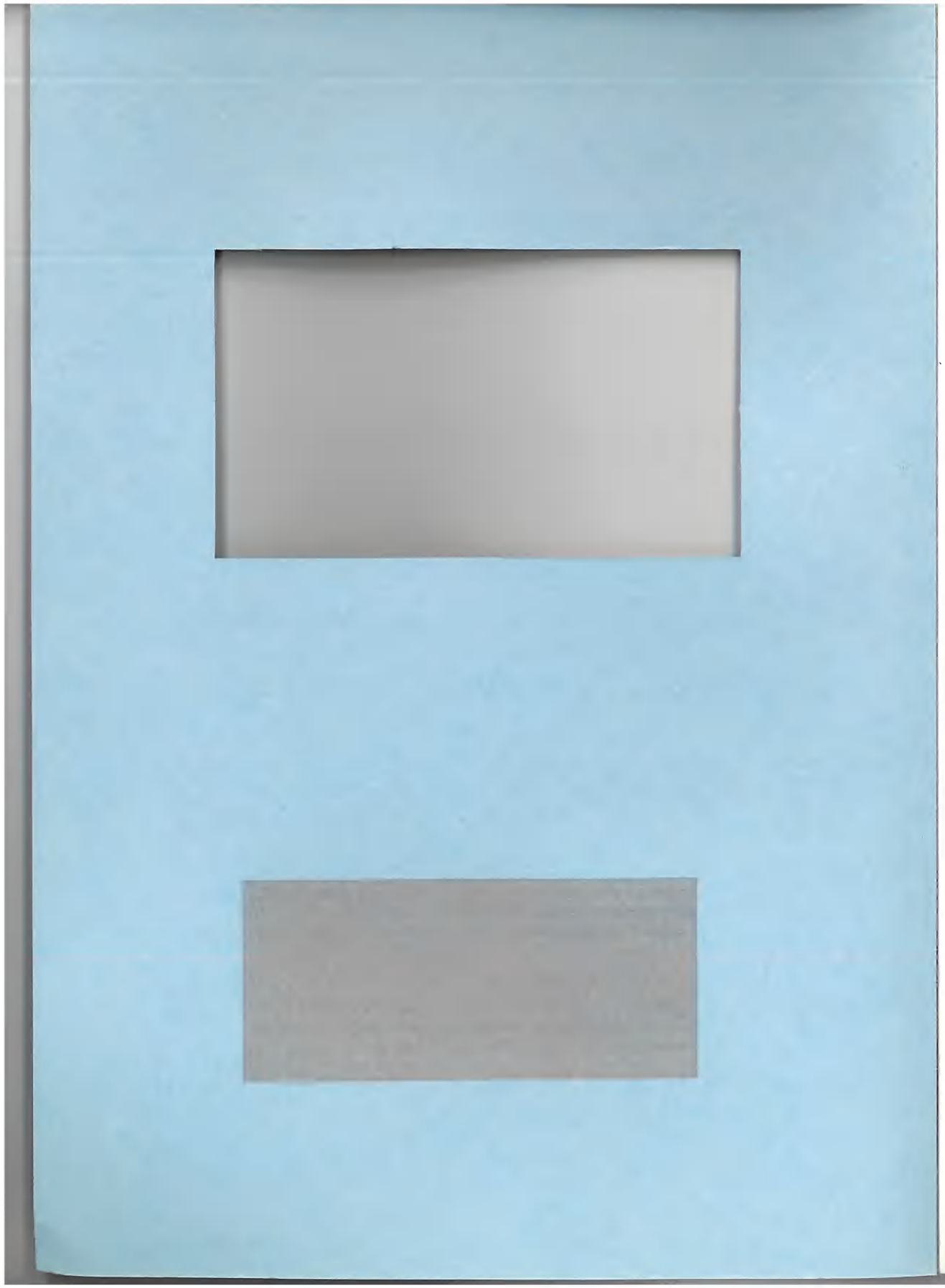

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Abstract

Linear mode conversion between electromagnetic and electrostatic waves is represented for thermal plasmas with an inhomogeneity in one dimension. The wave equations are derived in fluid approximation and solutions inside and outside of the conversion region are discussed. The boundary value problem of mode conversion is solved and the conversion rate is calculated explicitly.

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1.1 Introduction

Electromagnetic wave reflection from plane surface layers is commonly treated by the dielectric model[1,2] of cold electrons and discontinuous boundaries. This simple model, however, does not completely describe plasmas where both thermal motion and continuous density transitions are essential features. The purpose of this study is a systematic analysis of these additional aspects in terms of a fluid description.

The fluid theory of electromagnetic and electrostatic waves is well known for homogeneous equilibrium conditions[3], but there exists no generally accepted fluid model for the inhomogeneous case. Wave equations for inhomogeneous equilibria have been derived both from fluid theories[4,5] and from the Vlasov equation[6,7], but there is disagreement on various terms containing gradients of the space-dependent coefficients. Here we present a consistent derivation from the Vlasov-Maxwell theory and carefully discuss the limitations of the model.

Solutions to the wave equation for continuous transition layers are well known for s-polarized light (electric field perpendicular to the plane of incidence)[4,8-11] and convergence to the discontinuous case has been shown there for layers smaller than about 1/10 of the wavelength[12]. Collisional absorption[13-15] can also be regarded as well understood provided the collision frequency is known. We therefore restrict our attention to p-polarization (electric field parallel to the plane of incidence), Fig. 1, and consider only the collisionless limit. Under these conditions mode conversion between electromagnetic and electrostatic waves has been described for smooth gradients in cold ($\beta^2 = 0$)[4,16-18] and low-temperature ($\beta^2 \ll 1$)[5,6, 19,20] plasmas and for intermediate gradients in cold plasmas[21]. Here we extend the discussion to the complete range of scale lengths and temperatures. We present solutions describing continuous transitions from smooth to thin surface

layers. Absorption at smooth gradients has been found to be independent of temperature[18,19]. We shall see that for thin layers this does not hold and finite temperatures can lead to complete conversion between the waves. For the high-temperature regime a relativistic fluid model is discussed.

The theory of linear mode conversion has important applications in the optics of ionospheric[8], solid-state[22,23] and laser plasmas[24-28]. In the latter case detailed measurements of absorption curves have recently been reported[29].

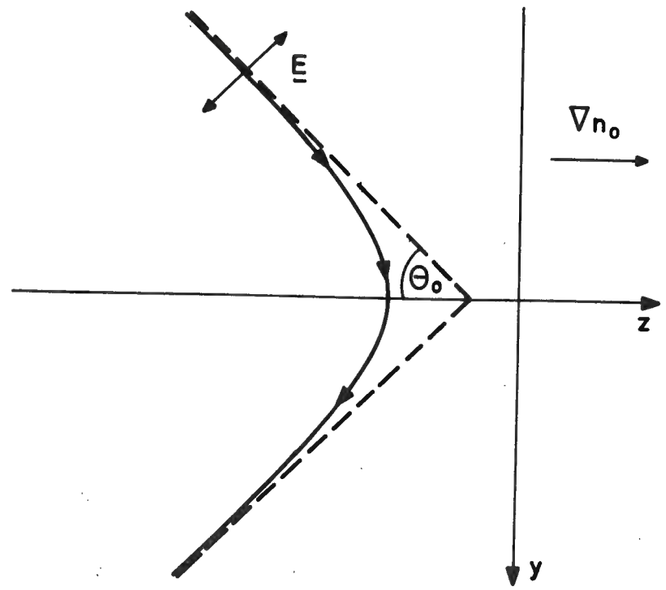


Fig. 1: Oblique incidence of a p-polarized electromagnetic wave. The electron density n_0 increases in the positive z-direction and the y-axis is chosen to lie in the plane of incidence

1.2 Summary of results

In Sec. 2 we are concerned with the discussion of general aspects of the fluid model. For a nonuniform electron fluid spatially varying in one dimension we derive the wave equations

$$B_x'' - \frac{\varepsilon'}{\varepsilon - \alpha_0^2\beta^2} B_x' + k_0^2(\varepsilon - \alpha_0^2) B_x = -\alpha_0\beta^2 \frac{\varepsilon'}{\varepsilon - \alpha_0^2\beta^2} E_z' \quad (1.1)$$

$$\beta^2 E_z'' + k_0^2(\varepsilon - \alpha_0^2\beta^2) E_z = k_0^2(1-\beta^2)\alpha_0 B_x \quad (1.2)$$

from the Vlasov-Maxwell theory. The derivation is based on the assumption that the thermal velocity and the Debye length are small compared with the phase velocity and the inhomogeneity scale length respectively. These constraints are expressed by the conditions

$$|\varepsilon| \ll 1 \quad ; \quad (\beta/k_0 L)^{2/3} \ll 1 \quad (1.3)$$

It is demonstrated that additional gradient terms in the wave equations as introduced by several authors[4,5,7] cannot be consistently retained within the fluid approximation (1.3).

The wave equations(1.1), (1.2) are shown to have WKB solutions of the form

$$E_z = \frac{\sin\theta^{em}}{(\varepsilon - \alpha_0^2)^{1/4}} C^{em,\pm} \exp\left\{\pm i k_0 \int \sqrt{\varepsilon - \alpha_0^2}\, dz\right\} \quad (1.4)$$

$$B_x = (\varepsilon/\alpha_0) E_z$$

and

$$E_z = \frac{\cos\theta^{es}}{(\varepsilon/\beta^2 - \alpha_0^2)^{1/4}} C^{es,\pm} \exp\left\{\pm i k_0 \int dz \sqrt{\frac{\varepsilon}{\beta^2} - \alpha_0^2}\right\} \quad (1.5)$$

$$B_x = 0$$

describing electromagnetic and electrostatic waves. In this solution the longitudinal polarization of the electrostatic wave (1.5) is correctly propagated along the ray. We note that this result cannot be obtained in the electrostatic approximation[6] currently used.

For linear density profiles we derive an exact solution to (1.1), (1.2) in the form of a convergent series expansion around the singular point

$$\epsilon - \alpha_0^2 \beta^2 = 0$$

with a general recurrence relation for the coefficients. We find

$$\begin{pmatrix} B_x \\ E_z \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & (1-\beta^2)\alpha_0 \left(\frac{k_0 L}{\beta}\right)^{2/3} \end{pmatrix} \sum_{n=0}^{\infty} \underline{Z}_n(\lambda) \left\{^{\lambda+n} \right. \qquad (1.6)$$

with

$$\lambda \in \{0, 1, 2\}$$

and where

$$\underline{Z}_0(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \underline{Z}_0(1) = c_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \underline{Z}_0(2) = c_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{Z}_1(\lambda) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{Z}_2(0) = c_1 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ p/2 \end{pmatrix}; \quad \underline{Z}_2(1) = c_3 \begin{pmatrix} 0 \\ p/6 \end{pmatrix}; \quad \underline{Z}_2(2) = c_4 \begin{pmatrix} 0 \\ 1/12 \end{pmatrix}$$

$$\underline{Z}_n(\lambda) = \underline{\underline{U}}_1 \cdot \underline{Z}_{n-3}(\lambda) + \underline{\underline{U}}_2 \cdot \underline{Z}_{n-2}(\lambda) \quad ; \quad n \geq 3$$

$$\underline{\underline{U}}_1 = \begin{pmatrix} \dfrac{\beta^2}{(\lambda+n)(\lambda+n-2)} & \dfrac{-p/(1-\beta^2)}{(\lambda+n)(\lambda+n-2)} \\ 0 & \dfrac{1}{(\lambda+n)(\lambda+n-1)} \end{pmatrix}$$

$$\underline{\underline{U}}_2 = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{(\lambda+n)(\lambda+n-1)} & \dfrac{p}{(\lambda+n)(\lambda+n-1)} \end{pmatrix}$$

The first orders of the series (1.6) have been calculated by Piliya[6]. Our result generalizes this approach to arbitrary orders and proves completeness and convergence of the solution. From this we can conclude that the solution of the thermal problem ($\beta^2 \neq 0$) is single-valued. The calculation of the absorption coefficient is therefore independent of the integration path in the complex plane as has been explicitly shown by Pert[19].

Energy conservation of the fluid equations is established from Poynting's theorem. We find the local conversation law

$$\nabla \cdot \left\{ \langle \underline{S} \rangle + \langle \underline{T} \rangle \right\} = 0 \qquad (1.7)$$

$$\langle \underline{S} \rangle = \frac{c}{8\pi} \, \mathrm{Re} \left\{ \underline{E}(\underline{x}) \times \underline{B}^*(\underline{x}) \right\}$$

$$\langle \underline{T} \rangle = \frac{c}{8\pi} \, \mathrm{Re} \left\{ i\beta^2 \underline{E}(\underline{x}) \, k_0^{-1} \nabla \cdot \underline{E}^*(\underline{x}) \right\}$$

and by integrating the global relation

$$A = \frac{\pi k_0 L \alpha_0^2}{(1-\alpha_0^2)^{1/2}} \left| \frac{B_x(0)}{C_0} \right|^2 \qquad (1.8)$$

between the absorption coefficient A defined in the wave zone and the magnetic field amplitude $|B_x(0)|$ at the electrostatic wave turning point. Action flow conservation is well-known for WKB modes[30,31]. However, (1.7) also applies to regions where the WKB solution fails. The expression (1.8) agrees with the cold plasma solution[18]. It therefore confirms a universal absorption curve independent of collisions and temperature[18,19]. We note, however, that in deriving (1.8) from (1.7) the scale length was assumed to be larger than the electrostatic wavelength. In the opposite limit the temperature will be important, as discussed below.

At the end of Sec. 2 we pose the boundary value problem of linear mode conversion in a form convenient for numerical calculations. It is shown how the solution to the boundary value problem can be constructed from the solution of an initial value problem with initial values given in the overdense region.

In Sec. 3 the conversion problem is treated in detail both for cold ($\beta^2 = 0$) and thermal ($\beta^2 \neq 0$) plasmas. In the cold case we consider the exactly solvable model of a linear density profile with adjoining regions of constant density. Here the absorption coefficient is explicitly obtained as

$$A = 4\pi q\, m_1 |s_1| \left\{ P_2'(2) + m_2 P_2(2) \right\}^2 / |C|^2 \qquad (1.9)$$

with

$$\mathrm{Re}(C) = \left\{ P_2'(2) + m_2 P_2(2) \right\} \left\{ P_1'(1) + \frac{1}{2}\left(\frac{1}{s_1} + \pi m_1 \right) P_2(1) \right\} -$$

$$- P_2'(1) \left\{ m_2 P_1(2) + P_1'(2) + \frac{q}{2} \frac{P_2(2)}{s_2} \right\} +$$

$$+ \frac{q}{2} P_2'(1) \ln \left| \frac{s_1}{s_2} \right| \left\{ P_2'(2) + m_2 P_2(2) \right\}$$

$$\mathrm{Im}(C) = \left\{ P_2'(2) + m_2 P_2(2) \right\} \left\{ m_1 P_1(1) - \frac{q}{2} \pi P_2'(1) \right\} -$$

$$- m_1 P_2(1) \left\{ m_2 P_1(2) + P_1'(2) + \frac{q}{2} \frac{P_2(2)}{s_2} \right\} +$$

$$+ \frac{q}{2} m_1 P_2(1) \ln \left| \frac{s_1}{s_2} \right| \left\{ P_2'(2) + m_2 P_2(2) \right\}$$

$$m_{1/2} = \left(\mp s_{1/2} \mp q \right)^{1/2} \quad ; \quad P_{1/2}^{(1)}(1/2) = P_{1/2}^{(1)}(s_{1/2})$$

where $P_1(s)$ and $P_2(s)$ are power series in s. The solution (1.9) is evaluated as a function of the gradient scale length. For smooth gradients ($|s_{1/2}| \gg 1$) the well-known resonance curve is found. However, absorption decreases substantially for $|s_{1/2}| \lesssim 1$ and is shown to approach zero in the large gradient limit ($|s_{1/2}| \ll 1$).

More generally, absence of absorption at large gradients follows from the Fresnel boundary conditions

$$[B_x] = 0 \quad ; \quad [E_y] = 0 \qquad (1.10)$$

independently of the special profile model. We therefore revise the usual derivation of these conditions[32] taking into account the singular behaviour of the electric field.

Most of the literature on resonance absorption in a cold plasma has been concerned with the model of an overall linear density variation[4,16,18]. It applies well to the smooth gradient case, but the limit to large gradients cannot be carried out there. The profile model treated here has already been discussed in Ref. 17 but only within the parameter range $|s_1| \gtrsim 0{,}6$, $s_2 \gg 1$, where the large gradient effect of decreasing absorption is absent. Finally, we note that our model only applies to those angles of incidence where the wave turning point lies within the large gradient layer. For the opposite case coupling to surface waves with high absorption values has been shown[21,33]. Even here, however, absorption disappears if the gradients are sufficiently large for the boundary conditions (1.10) to apply.

In the thermal case eqs. (1.1), (1.2) together with the appropriate boundary conditions have been numerically solved for an Epstein layer[12]

$$\varepsilon(z) = \varepsilon_1 - \frac{\varepsilon_1 - \varepsilon_2}{1 + \exp\{-z/\Delta\}} \qquad (1.11)$$

The well-known resonance curve is obtained for temperatures below

$$\beta^2 \approx 0.1 \qquad (1.12)$$

and scale lengths larger than

$$\bar{k}\Delta \approx 1 \quad ; \quad (\bar{k})^{-1} = \{k_1^{-2} + k_2^{-2}\}^{1/2} \tag{1.13}$$

For this smooth gradient case the field structure is calculated as a function of the angle of incidence and compared with analytic expressions both in the WKB and in the resonance region.

A continuous transition to thin layers is described and an asymptotic absorption curve is found if the layer width is small compared with the electrostatic wavelength:

$$|\xi_{1/2}| \ll 1 \quad ; \quad \xi_{1/2} = \beta^{-2/3}\, s_{1/2} \tag{1.14}$$

This asymptote is analytically described by solving the model of two semi-infinite homogeneous plasma regions with a common interface. We get

$$A = 1 - R$$

$$R = \frac{(a-b)^2 + (c-d)^2}{(a+b)^2 + (c+d)^2} \tag{1.15}$$

with

$$a = \sqrt{\tau}\,\{u\hat{u}\hat{w} - (1+\tau)\beta\alpha_1^2\hat{w}\}$$

$$b = \sqrt{\tau}\,\beta^2\alpha_1^2 w$$

$$c = (1+\tau)\beta\alpha_1^2\hat{u} - \tau\beta^2\alpha_1^2 u$$

$$d = w\hat{w}\hat{u}$$

$$w = (1-\alpha_1^2)^{1/2}\,, \quad \hat{w} = (1-\beta^2\alpha_1^2)^{1/2}$$

$$u = (1+\tau\alpha_1^2)^{1/2}\,, \quad \hat{u} = (1+\beta^2\tau\alpha_1^2)^{1/2}$$

$$\tau = \varepsilon_1 / |\varepsilon_2|\,, \quad \alpha_1^2 = \alpha_0^2/\varepsilon_1\,.$$

The solution (1.15) describes complete conversion

$$A = 1 \tag{1.16}$$

for

$$\tau = 0 \tag{1.17}$$

$$\alpha_1^2 = 1/(1+\beta^2)\,. \tag{1.18}$$

Condition (1.17) is interpreted formally as the limit of an infinitely high potential barrier with the boundary condition

$$\underline{E} = 0 \tag{1.19}$$

for the electric wave field. By this condition any reflected wave is excluded at the angle (1.18), where the incident electromagnetic ray is perpendicular to the electrostatic ray.

The thermal problem for smooth gradients has been treated both analytically[6,19] and numerically[5]. The purpose here is to give a detailed comparison of the complete numerical solution to the analytic theory. The discussion of the field structure in Ref. 5 is extended to angles which are off resonance. The case where the electrostatic wavelength gets larger than the layer thickness has not been considered so far. Modifications to the Fresnel reflection coefficient due to optical excitation of plasma waves are known for vacuum-metal interfaces[22,23]. However, no outgoing plasma wave is considered there and the effect of complete conversion is therefore absent.

In Sec. 4 we briefly discuss nonlinear, kinetic and relativistic aspects of mode conversion. We derive the wave-breaking condition for nonlinear stationary solutions to the cold plasma model. The amplitude on wave-breaking

$$\delta_{max} = (\delta_0 L)^{1/2} \quad ; \quad \delta_0 = \frac{q}{m\omega^2} |B_x(0)|^2$$

is found here to be lower than predicted from non-stationary solutions[34].

Linear Landau damping is analytically calculated. We find the damping argument

$$\Psi(z) = \int_0^z k_i \, dz' = \left(\frac{3}{2}\pi\right)^{1/2} \frac{k_0 L}{\beta} \exp\left\{\frac{3}{2}\frac{L}{z}\right\} \qquad (1.21)$$

which is valid outside the resonance ($\xi \leq -1$). For a fixed density it depends linearly on the parameter $k_0 L/\beta$.

Finally, the fluid model is generalized to the case of relativistic temperatures. The relativistic wave equations are obtained from the classical ones by the substitutions

$$\varepsilon \longrightarrow \frac{\varepsilon_T}{1 + T/m} \qquad (1.22)$$

$$\beta^2 \longrightarrow \frac{\beta_T^2}{1 + T/m}$$

With the non-relativistic and ultra-relativistic limits

$$\varepsilon_T = \begin{cases} 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{5}{2}\frac{T}{m}\right) & T/m \ll 1 \\ 1 - \frac{\omega_p^2}{\omega^2}\frac{m}{4T} & T/m \gg 1 \end{cases} \qquad (1.23)$$

$$\beta_T^2 = \begin{cases} 3\frac{T}{m}\left(1 - \frac{5}{2}\frac{T}{m}\right) & T/m \ll 1 \\ \frac{2}{3} & T/m \gg 1 \end{cases}$$

2.1 Fluid approximation

In the following a consistent derivation of the wave equations is presented. It is valid in the fluid approximation where the expansion parameter is

$$\delta = \max\{\delta_1, \delta_2\} \ll 1 \quad ; \quad \delta_1 = \frac{S}{S_{ph}} \quad ; \quad \delta_2 = \frac{\lambda_{De}}{L} \qquad (2.1)$$

The moments of the Vlasov equation are solved to the second order in δ. To this accuracy the resulting fluid equations are written in the form of a viscous fluid with a formal complex collision frequency. We proceed by discussing the general form of the dispersion relation within this model and find the term

$$(\beta/k_0)^2 \nabla(\nabla \cdot \underline{E}_1) \quad ; \quad \beta^2 = \frac{3T_0}{mc^2} \quad ; \quad k_0 = \frac{\omega}{c} \qquad (2.2)$$

to be the only thermal contribution to the wave equations which is consistent with the fluid approximation (2.1).

We consider the linearized Vlasov-Maxwell equations for perturbations of the electron equilibrium:

$$\{\partial_t + \underline{v}\cdot\nabla + \frac{q}{m}(\underline{E}_0 + \frac{1}{c}\underline{v}\times\underline{B}_0)\cdot\partial_{\underline{v}}\} f_1 + \frac{q}{m}(\underline{E}_1 + \frac{1}{c}\underline{v}\times\underline{B}_1)\cdot\partial_{\underline{v}} f_0 = 0 \qquad (2.3)$$

$$\nabla\times\underline{B}_1 = \frac{1}{c}\partial_t \underline{E}_1 + \frac{4\pi}{c} q \int d^3\underline{v}\, \underline{v} f_1 \qquad (2.4)$$

$$\nabla\times\underline{E}_1 = -\frac{1}{c}\partial_t \underline{B}_1 \qquad (2.5)$$

$$\nabla\cdot\underline{E}_1 = 4\pi q \int d^3\underline{v}\, f_1 \quad ; \quad \nabla\cdot\underline{B}_1 = 0 \qquad (2.6)$$

On the equilibrium we impose the conditions

$$f_0 = f_0(\underline{x}, v^2) \quad ; \quad \underline{B}_0 = 0 \qquad (2.7a)$$

Then from the first moment of the Vlasov equation we get

$$q n_0 \underline{E}_0 = \nabla p_0 \qquad (2.7b)$$

with

$$n_0 = \int d^3_{\underline{v}}\, f_0 \quad ; \quad p_0 = n_0 T_0 = \frac{m}{3} \int d^3_{\underline{v}}\, v^2 f_0$$

For any equilibrium solution satisfying (2.7) the pressure gradient is balanced by an ambipolar electric field, while magnetic fields and plasma flow are neglected.

Looking for solutions with harmonic time dependence

$$f_1 \propto e^{-i\omega t} \quad ; \quad \underline{E}_1 \propto e^{-i\omega t} \quad ; \quad \underline{B}_1 \propto e^{-i\omega t} \tag{2.8}$$

(2.3) - (2.5) reduce to

$$\left(1 + \frac{1}{-i\omega}\, \underline{v}\cdot\nabla + \frac{q}{-i\omega m}\, \underline{E}_0 \cdot \partial_{\underline{v}}\right) f_1 + \frac{q}{-i\omega m}\, \underline{E}_1 \cdot \partial_{\underline{v}} f_0 = 0 \tag{2.9}$$

$$\nabla \times \underline{B}_1 = -i k_0 \left(\underline{E}_1 + \frac{4\pi i}{\omega}\, q \int d^3_{\underline{v}}\, \underline{v}\, f_1\right) \tag{2.10}$$

$$\nabla \times \underline{E}_1 = i k_0 \underline{B}_1 \tag{2.11}$$

while (2.6) is automatically satisfied for every C^1 solution f_1, $\underline{E}_1$, $\underline{B}_1$ of (2.9) - (2.11) according to the integral conditions

$$\int_V d^3_{\underline{x}}\, \nabla\cdot\underline{B}_1 = \int_S d\underline{S}\cdot\underline{B}_1 = \frac{1}{i k_0} \int_{\partial S} d\underline{s}\cdot\underline{E}_1 = 0$$

$$\int_V d^3_{\underline{x}} \left(\nabla\cdot\underline{E}_1 - 4\pi q \int d^3_{\underline{v}}\, f_1\right) = \int_S d\underline{S}\cdot\underline{E}_1 - \int_V d^3_{\underline{x}}\, 4\pi q \int d^3_{\underline{v}}\, f_1 = \frac{1}{-i k_0} \int_{\partial S} d\underline{s}\cdot\underline{B}_1 = 0$$

for arbitrary volumes V.

Taking the moments of the Vlasov equation (2.9), we get

$$n_1 + \frac{\delta}{-i\omega}\, \partial_{x^i} M_1^i = 0 \tag{2.12}$$

$$M_1^{ij\cdots m} + \frac{\delta}{-i\omega}\, \partial_{x^n} M_1^{ij\cdots mn} - \frac{\delta q}{(-i\omega m)} (E_0 M_1)^{(ij\cdots m)} - \frac{q}{(-i\omega m)} (E_1 M_0)^{(ij\cdots m)} = 0$$

with

$$n_1 = \int d^3_{\underline{v}}\, f_1$$

$$M^{ij\cdots m} = \int d^3_{\underline{v}}\, v^i v^j \cdots v^m f$$

$$(E\,M)^{(ij\cdots m)} = E^i M^{j\cdots m} + \left\{\begin{array}{l}\text{all terms obtained by}\\ \text{cyclic permutations of}\\ \text{indices}\end{array}\right.$$

and where we have denoted by δ all small terms arising from

$$\frac{1}{-i\omega}\, \underline{v}\cdot\nabla \sim \max\left\{\frac{s k}{\omega}, \frac{s}{\omega L}\right\} = \delta$$

$$\frac{q}{-i\omega m}\, \underline{E}_0\cdot\partial_{\underline{v}} = \frac{1}{-i\omega}\, \frac{T_0}{m}\, \frac{\nabla p_0}{p_0}\cdot\partial_{\underline{v}} \sim \frac{s}{\omega L} \leq \delta$$

$$s = \sqrt{3}\, \omega \lambda_{De} = \left(\frac{3 T_0}{m}\right)^{1/2}$$

The wave number k and the scale length L are defined here by the maximum values of the gradients occurring over a wavelength and over the inhomogeneity region respectively. Eliminating higher moments and keeping terms up to the order δ^2, we get from (2.12)

$$M_1^i = \frac{q}{-i\omega m}\, n_0 E_1^i + \tag{2.13}$$

$$+ \delta^2 \frac{q}{\omega^2 m} \left\{ E_0^i\, \partial_{x^k} M_1^k + \partial_{x^i} (E_0 M_1)^{(ij)} + \frac{1}{i\omega}\, \partial_{x^i} \partial_{x^k} (E_1 M_0)^{(ijk)} \right\}$$

If we introduce conventional hydrodynamic variables

$$n_0 \underline{V}_1 = \underline{M}_1 \quad ; \quad \underline{\underline{P}}_1 = p_1 \underline{\underline{I}} + \underline{\underline{\Pi}}_1 \quad ; \quad p_1 = \frac{1}{3} \mathrm{Tr}\,(\underline{\underline{P}}_1)$$

we find from (2.12), (2.13)

$$-i\omega n_1 + \nabla\cdot(n_0 \underline{V}_1) = 0 \tag{2.14a}$$

$$-i\omega m n_0 \underline{V}_1 = q(n_0 \underline{E}_1 + n_1 \underline{E}_0) - \nabla p_1 - \nabla\cdot\underline{\underline{\Pi}}_1 \tag{2.14b}$$

with

$$p_1 = \frac{5}{3} T_0 n_1 - \frac{5}{3} \frac{T_0}{i\omega} \underline{v}_1 \cdot \nabla n_0 + \frac{1}{i\omega} \underline{v}_1 \cdot \nabla p_0 \qquad (2.15a)$$

$$\underline{\underline{\Pi}}_1 = -\eta \underline{\underline{W}}_1 \quad ; \quad \eta = \frac{p_0}{-i\omega} \qquad (2.15b)$$

$$W_1^{ij} = \partial_{xi} v_1^j + \partial_{xj} v_1^i - \frac{2}{3} \nabla \cdot \underline{v}_1 \delta^{ij}$$

where $\underline{\underline{I}}$ denotes the unit tensor with components δ^{ij}, $Tr(\ )$ the trace, and where the expansion parameter δ has been omitted.

The fluid model (2.14), (2.15) has the form of a viscous fluid with an adiabatic energy law. The traceless part $\underline{\underline{\Pi}}_1$ of the pressure tensor agrees with the usual viscosity tensor[35] if the collision frequency there is replaced by $-i\omega$. The isotropic pressure p_1 satisfies the adiabatic law for an ideal gas with three degrees of freedom

$$\frac{d}{dt} S = 0 \quad ; \quad S = const + \frac{3}{2} \ln \left(\frac{p}{n^{5/3}} \right) \quad ; \quad \frac{d}{dt} = \partial_t + \underline{v} \cdot \nabla$$

applied along the flow lines. Comparing the change in thermal energy density $\sim \omega p_1$ with the divergence of the heat flux

$$\frac{k S_{gr} p_1}{\omega p_1} \sim \frac{S_{gr}}{S_{ph}} \sim \left(\frac{S}{S_{ph}} \right)^2 \leq \delta^2$$

heat conduction is seen to contribute only to higher orders of the expansion.

Solving (2.14), (2.15) to the order δ^2 for the current density

$$\frac{4\pi i}{\omega} q n_0 \underline{v}_1 = -\frac{\omega_p^2}{\omega^2} \underline{E}_1 + \frac{1}{m\omega^2} \left\{ \nabla \partial_{xi} \left(T_0 \frac{\omega_p^2}{\omega^2} E_1^i \right) + \right.$$

$$\left. + \partial_{xi} \left(T_0 \frac{\omega_p^2}{\omega^2} \right) \partial_{xi} \underline{E}_1 + \partial_{xi} \left(T_0 \frac{\omega_p^2}{\omega^2} \right) \nabla E_1^i - \right.$$

$$\left. - T_0 \frac{\nabla p_0}{p_0} \partial_{xi} \frac{\omega_p^2}{\omega^2} E_1^i \right\} \quad ; \quad \omega_p^2 = \frac{4\pi q^2 n_0}{m} \qquad (2.16)$$

we get together with (2.9), (2.10)

$$\frac{1}{-ik_0} \nabla \times \underline{B}_1 = \varepsilon \underline{E}_1 + \frac{1}{m\omega^2} \left\{ \nabla \partial_{xi} T_0 (1-\varepsilon) E_1^i + \partial_{xi} T_0 (1-\varepsilon) \nabla E_1^i + \right.$$

$$\left. + \partial_{xi} T_0 (1-\varepsilon) \partial_{xi} \underline{E}_1 - T_0 \frac{\nabla p_0}{p_0} \partial_{xi} (1-\varepsilon) E_1^i \right\}$$

$$\frac{1}{ik_0} \nabla \times \underline{E}_1 = \underline{B}_1 \quad ; \quad \varepsilon = 1 - \frac{\omega_p^2}{\omega^2} \qquad (2.17)$$

We now qualitatively discuss (2.17).
Taking the divergence there it will be of the general form

$$k\varepsilon + \frac{1}{L} + \delta_1^2 k (1-\varepsilon) \left\{ 1 + (kL)^{-1} + (kL)^{-2} + (kL)^{-3} \right\} = 0$$

or equivalently

$$\varepsilon + (kL)^{-1} + (1-\varepsilon) \left\{ \delta_1^2 + (kL)^{-1} (\delta_1^2 + \delta_1 \delta_2 + \delta_2^2) \right\} = 0 \qquad (2.18)$$

In order to be consistent with the fluid approximation, (2.18) can only be solved to the order δ^2. All space dependent terms $\propto \delta^2 \varepsilon$ as well as all gradient terms $\propto \delta^2/kL$ therefore have to be dropped, and the temperature

$$T_0(\underline{x}) \approx T_0(x)\Big|_{\underline{x}=0} \left(1 + \frac{\nabla T_0}{T_0} \cdot \underline{x} \right) \sim T_0(x)\Big|_{\underline{x}=0} (1+\varepsilon)$$

has to be regarded as constant. Then (2.18) gives

$$\varepsilon + (kL)^{-1} + \delta_1^2 = 0 \qquad (2.19)$$

Dropping the same terms in the initial equations (2.17) and assuming non-relativistic temperatures

$$\beta^2 << 1$$

we find

$$i \nabla \times \underline{B}_1 = k_0 \left(\varepsilon \underline{E}_1 + \left(\frac{\beta}{k_0}\right)^2 \nabla \nabla \cdot \underline{E}_1 \right) \qquad (2.20a)$$

$$\nabla \times \underline{E}_1 = i k_0 \underline{B}_1 \qquad (2.20b)$$

We now solve (2.19).

(1) Cold plasma

If the dielectric constant is much larger than the expansion parameter:

$$|\varepsilon| >> \delta_1^2 \qquad (2.21a)$$

(2.19) has the solution

$$k = \frac{1}{\varepsilon L} \qquad (2.21b)$$

Inserting (2.21b) into (2.21a) the solution proves to be consistent for

$$|\varepsilon| >> (\beta/k_0 L)^{2/3} \qquad (2.22a)$$

We note that for a complex dielectric constant the cold plasma model is consistent even at the critical density if

$$|Jm(\varepsilon)| >> (\beta/k_0 L)^{2/3} \qquad (2.22b)$$

Equating the wave number (2.21b) to the wave number in the homogeneous region

$$k = k_0 \sqrt{\varepsilon}$$

we find

$$(k_0 L)^{2/3} \varepsilon = 1 \quad ; \quad k = k_0 / (k_0 L)^{1/3} \qquad (2.23)$$

(2) Thermal plasma: $|\varepsilon| \lesssim \delta_1^2$

For large and small density gradients (2.19) leads to

$$|\varepsilon| << (\beta/k_0 L)^{2/3} << 1 : \quad k = \frac{1}{L} \left(\frac{k_0 L}{\beta}\right)^{2/3} \qquad (2.24)$$

$$(\beta/k_0 L)^{2/3} << |\varepsilon| << 1 : \quad k = k_0 \sqrt{\varepsilon/\beta^2} \qquad (2.25)$$

Equating both solutions we find

$$(k_0 L/\beta)^{2/3} \varepsilon = 1 \quad ; \quad k = \frac{k_0}{\beta} \left(\frac{\beta}{k_0 L}\right)^{1/3} \qquad (2.26)$$

Finally, we specialize eqs. (2.20a), (2.20b) to the case of a p-polarized wave

$$\underline{E}_1 = (0, E_{y(z)}, E_{z(z)}) \exp i(k_0 \alpha_0 y - \omega t) \qquad (2.27)$$

$$\underline{B}_1 = (B_{x(z)}, 0, 0) \exp i(k_0 \alpha_0 y - \omega t)$$

reflected from a plane layered medium with the density gradient in the z-direction

$$n_o = n_o(z)$$

With the ansatz (2.27) we get from (2.20)

$$\begin{aligned} k_o B_x &= k_o \alpha_o E_z + i E_y' \\ i B_x' &= k_o \left\{ \varepsilon E_y + \left(\frac{\beta}{k_o}\right)^2 \left(-\alpha_o^2 k_o^2 E_y + i \alpha_o k_o E_z' \right) \right\} \\ \alpha_o B_x &= \varepsilon E_z + \left(\frac{\beta}{k_o}\right)^2 \left(E_z'' + i \alpha_o k_o E_y' \right) \end{aligned} \tag{2.28}$$

where the prime denotes differentiation with respect to z. Eliminating E_y and E_y' from (2.28) a pair of coupled wave equations is obtained:

$$\begin{aligned} B_x'' - \frac{\varepsilon'}{\varepsilon - \alpha_o^2 \beta^2} B_x' + k_o^2 (\varepsilon - \alpha_o^2) B_x &= -\alpha_o \beta^2 \frac{\varepsilon'}{\varepsilon - \alpha_o^2 \beta^2} E_z' \\ \beta^2 E_z'' + k_o^2 (\varepsilon - \alpha_o^2 \beta^2) E_z &= k_o^2 \alpha_o (1 - \beta^2) B_x \end{aligned} \tag{2.29}$$

We rewrite eqs. (2.29) as

$$\begin{aligned} H'' - \frac{\varepsilon_o'}{\varepsilon_o} H' + k_o^2 \varepsilon_o H &= \alpha_o (1 - \beta^2) k_o^2 \varepsilon_o E_z \\ \beta^2 E_z'' + k_o^2 (\varepsilon_o - \alpha_o^2 \beta^2 (1 - \beta^2)) E_z &= k_o^2 \alpha_o (1 - \beta^2) H \\ H = B_x - \alpha_o \beta^2 E_z \quad ; \quad \varepsilon_o &= \varepsilon - \alpha_o^2 \beta^2 \end{aligned} \tag{2.30}$$

which is the form of the coupled wave equations used by Piliya[6]. In the special case of a cold plasma, $\beta^2 = 0$, from (2.29) a single differential equation for the magnetic field

$$B_x'' - \frac{\varepsilon'}{\varepsilon} B_x' + k_o^2 (\varepsilon - \alpha_o^2) B_x = 0 \tag{2.31}$$

$$E_z = \frac{\alpha_o B_x}{\varepsilon}$$

follows.

2.2 Range of applicability

In this section the main assumptions underlying the fluid model will be discussed in detail. Quantitative estimates are given using parameters of typical laser plasma experiments (light wavelength: 1.06 μm ; light frequency: 1.8×10^{15} Hz ; critical density: 10^{21} cm^{-3} ; temperature: 3 keV ; gradients: $k_0 L = 8$).

Vlasov approximation: The Vlasov theory is applicable to collisionless plasmas with a large number of particles N within the Debye volume. For an electron-proton plasma we have[3]

$$N = \frac{4}{3}\pi \lambda_D^3 2n_0 \quad ; \quad \lambda_D = \left(\frac{T_0}{8\pi q^2 n_0}\right)^{1/2} = 4.9 \left(\frac{T_0}{n_0}\right)^{1/2} \frac{1}{(^\circ K cm)^{1/2}}$$

At the critical density N is large for both typical electron and ion temperatures:

$$N = 6454 \quad ; \quad T_0 = 3 \ keV$$

$$N = 458 \quad ; \quad T_0 = 500 \ eV$$

The electron-ion collision frequency[35]

$$\nu = \frac{4}{3}\sqrt{2\pi}\,\frac{q^4}{\sqrt{m}}\,\ln\Lambda\,\frac{n_0}{T_0^{3/2}} \quad ; \quad \Lambda = \frac{\lambda_0}{\hbar/(ms)}$$

is typically of the order

$$\nu = 27\,\frac{n_0}{T_0^{3/2}}\,(^\circ K)^{3/2}\,cm^3\,Hz = 1.3 \times 10^{11}\ Hz \qquad (2.32)$$

$$\ln\Lambda = 7.4$$

Using (2.25), (2.26) the time between subsequent collisions ν^{-1} proves

to be much larger than the characteristic time scale of the problem $(k s_{gr})^{-1}$:

$$\frac{1}{\nu} >> \frac{1}{k s_{gr}} \sim \left(\frac{s}{\sqrt{\varepsilon}\,\omega}\right)\left(\frac{1}{\sqrt{\varepsilon}\,s}\right) \sim \frac{1}{\omega}\left(\frac{k_0 L}{\beta}\right)^{2/3}$$

$$\frac{\nu}{\omega} = 0.7 \times 10^{-4} \quad ; \quad \left(\frac{\beta}{k_0 L}\right)^{2/3} = 0.06$$

Fluid approximation: All terms neglected by the fluid approach are of second order in the expansion parameter δ. Using (2.1), (2.24), (2.25) the condition

$$\delta^2 << 1$$

can be expressed by

$$|\varepsilon| << 1 \qquad (2.33a)$$

and

$$\left(\frac{\beta}{k_0 L}\right)^{2/3} << 1 \qquad (2.33b)$$

Denoting the density where the dielectric constant vanishes the critical density n_c (2.33a) gives

$$\left|\frac{n_c - n_0}{n_c}\right| << 1$$

The departure from the critical density should therefore be less than about 10%. Condition (2.33b) gives a lower limit on the scale length. This limit is usually much smaller than the vacuum wavelength:

$$L >> \beta/k_0 \sim 2 \times 10^{-6}\ cm$$

The fluid model can only be applied to the problem of mode conversion if (2.33a) is satisfied within the whole conversion region. For a linearly varying density the change $\Delta\varepsilon$ in the dielectric constant over the resonance width (2.24) and the first subsequent wavelength (2.25) is about

$$\Delta\varepsilon \sim 4\left(\frac{\beta}{k_0 L}\right)^{2/3} \sim 0.24 \qquad (2.34)$$

The factor of the order of 4 makes (2.33a) more restrictive than (2.33b).

Static equilibrium: Plasma flow modifies the dispersion relation (2.19) by terms of the order (v_0/s_{ph}). The static approximation (2.7a) will therefore be valid if

$$v_0/s_{ph} \ll \max\{|\varepsilon|, \delta_1^2\} \qquad (2.35)$$

Inserting (2.25) into (2.35) we get for the thermal case

$$v_0 \ll \delta_1^2 s_{ph} \sim \varepsilon s_{ph} \sim \sqrt{\varepsilon}\, s \sim s_{gr} \qquad (2.36)$$

Condition (2.36) shows that the flow velocity is assumed to be small compared with the group velocity of the electrostatic wave. From (2.26), (2.36) we obtain for the ion Mach number the restriction

$$M \ll \sqrt{\varepsilon}\sqrt{\frac{m_i}{m}} \sim \left(\frac{\beta}{k_0 L}\right)^{1/3}\sqrt{\frac{m_i}{m}} = 11 \qquad (2.37)$$

which is usually well satisfied even for supersonic flow. Plasma flow has been introduced as a damping mechanism of the resonance field[36]. Comparing the saturation time of the resonance field in a cold plasma

$$t_1 \sim \nu^{-1}$$

with the time delay of a streaming electron in the resonance

$$t_2 \sim (k v_0)^{-1}$$

we find

$$\frac{t_1}{t_2} \sim \frac{k v_0}{\omega}\,\frac{\omega}{\nu} \sim \frac{v_0}{s_{ph}}\,\frac{1}{|\varepsilon|}$$

For

$$\frac{t_1}{t_2} \ll 1$$

the damping by flow should be negligible in agreement with the general condition (2.35).

Non-magnetized plasma: Magnetic field corrections to the dispersion relation (2.19) are of the order

$$\frac{\omega_c}{\omega} \quad ; \quad \omega_c = \frac{q B_0}{mc}$$

The assumption (2.7a) of a non-magnetized plasma is therefore valid for

$$\nu/\omega \gg \omega_c/\omega \; ; \quad B_0 \ll 5.7\times 10^{-11}\,\nu\,\frac{\text{K Gauß}}{\text{Hz}} = 7.4 \text{ K Gauß}$$

in the cold plasma (Sec. 4.2) and correspondingly for

$$\left(\frac{\beta}{k_0 L}\right)^{2/3} \gg \omega_c/\omega \; ; \quad B_0 \ll 6.5 \text{ M Gauß}$$

in the thermal plasma. In agreement with this estimate corrections to the absorption curve in a magnetized plasma have been calculated for magnetic fields $\gtrsim 1$ /MGauß/ [7].

Linear approximation: In the cold plasma model (Sec.4.1) linearization is valid if the electron displacement $\delta(z_0, t)$ slowly changes with the initial position z_0:

$$\left| \frac{\partial \delta(z_0, t)}{\partial z_0} \right| << 1 \qquad (2.38)$$

Inserting the solution (4.14) for $\delta(z_0, t)$, we find this condition to be satisfied for all z_0 and t if

$$k\,\delta_{max} << 1 \; ; \quad \delta_{max} = \frac{\omega}{\nu} \frac{q \alpha_0 B_c}{m \omega^2} \; ; \quad k = \frac{1}{\frac{\nu}{\omega} L} \qquad (2.39)$$

or equivalently if

$$\frac{v_{os}\, k}{\omega} << 1 \; ; \quad v_{os} = \omega\, \delta_{max} \qquad (2.40)$$

where δ_{max} is the maximum electron displacement and k the wave number (2.21).

For the thermal plasma model a similar restriction will hold. As an estimate we substitute

$$\frac{\nu}{\omega} \longrightarrow \left(\frac{\beta}{k_0 L}\right)^{2/3}$$

in (2.39) and put the maximum acceptable value of $k\delta_{max}$ one order of magnitude smaller than unity:

$$k\,\delta_{max} \leq 0.1 \; ; \quad \delta_{max} = \left(\frac{k_0 L}{\beta}\right)^{2/3} \frac{q \alpha_0 B_c}{m \omega^2} \; ; \quad k = \frac{1}{(\beta / k_0 L)^{2/3}} \frac{1}{L} \qquad (2.41)$$

Then, using the energy conservation law (2.99), we get for the incident light intensity the threshold

$$S = \frac{c}{8\pi} |C_0|^2 \leq 1.21 \times 10^{-14} \frac{\omega^2 (\beta^2)^{4/3} (k_0 L)^{1/3}}{A (1 - \alpha_0^2)} \frac{W s^2}{cm^2} \qquad (2.42)$$

with a typical value of

$$S \leq 7.5 \times 10^{14} \frac{W}{cm^2} \; ; \quad A = 0.5 \; ; \quad \alpha_0^2 = \frac{0.5}{(k_0 L)^{2/3}} \qquad (2.43)$$

The absolute value (2.43) depends quadratically on the numerical constant 0.1 introduced in (2.43). It therefore represents only a rough

estimate on the validity of linear theory. The scaling of the intensity threshold given by (2.42) shows a quadratic dependence on the frequency for $k_0 L$ kept fixed. The threshold is therefore lower by a factor of 10^2 for CO_2 laser systems.

Landau damping: The fluid description neglects damping of the waves by wave-particle interaction. Calculating the effect of linear Landau damping (Sec. 4.3) we find appreciable damping only outside the conversion region (2.34).

Non-relativistic approximation: Relativistic corrections at high oscillation velocities are nonlinear. Comparing them with the nonlinear terms (2.39), (2.40) of the classical theory we have

$$\frac{(v_{os}/c)^2}{(v_{os} k/\omega)^2} = \left(\frac{k_0}{k}\right)^2 = \begin{cases} [(\nu/\omega)\, k_0 L]^2 & \text{cold plasma} \\ \left[\left(\frac{\beta}{k_0 L}\right)^{2/3} k_0 L\right]^2 & \text{thermal plasma} \end{cases} \qquad (2.44)$$

For steep gradients this ratio is small and the wave will break before relativistic terms become important. On the other hand, for smooth gradients relativistic effects could dominate. Numerical calculations[37] showed the absence of large relativistic effects in the wave-breaking condition for typical plasma parameters.

Relativistic corrections at high temperatures are of the order β^2 (Sec. 4.4). They can be neglected in the dispersion relation (2.19) if

$$\beta^2 << \delta_1^2 \sim \left(\frac{\beta}{k_0 L}\right)^{2/3} << 1$$

Then the non-relativistic approximation is valid for

$$\beta^2 \ll \min\left\{1, \frac{1}{k_o L}\right\} \tag{2.45}$$

At characteristic temperatures, $\beta^2 = 0.018$, condition (2.45) is well satisfied, except for very large density scale lengths.

2.3.1 WKB solutions

The WKB (Wentzel[38], Kramers[39], Brillouin[40]) theory determines the normal modes to linear systems of differential[41] and integral[31] equations with slowly varying inhomogeneities. Its application to the system (2.32) provides electromagnetic and electrostatic wave solutions valid outside the region of mode conversion.

We consider the dielectric constant of the form

$$\varepsilon = \varepsilon(z/L) \quad ; \quad (k_o L)^{-1} = \tau \ll 1$$

Introducing $u = z/L$ as the independent variable, eqs. (2.32a), (2.32b) give

$$\tau^2\left(B_x'' - \frac{\varepsilon_o'}{\varepsilon_o} B_x'\right) + (\varepsilon - \alpha_o^2) B_x = -\tau^2 \alpha_o \beta^2 \frac{\varepsilon_o'}{\varepsilon_o} E_z' \tag{2.47a}$$

$$\tau^2 \beta^2 E_z'' + \varepsilon_o E_z = \alpha_o (1-\beta^2) B_x \tag{2.47b}$$

where the prime denotes differentiation with respect to u. In the limit $\tau \longrightarrow 0$ the medium can be regarded as locally homogeneous and the phase of the wave in any point u is proportional to τ^{-1}:

$$\Delta\phi \approx k_z \Delta z = \frac{k_z}{k_o} \frac{\Delta u}{\tau}$$

For small but finite values of τ we seek the phase in the form of a series expansion

$$\phi(u,\tau) = \tau^{-1} \sum_{n=0}^{\infty} \phi_n(u)\, \tau^n$$

Retaining only the leading term in the exponent, the WKB ansatz can be written as

$$B_x(u,\tau) = \left(\sum_{n=0}^{\infty} b_n(u)\,\tau^n\right) \exp\left(i\,\phi_0(u)/\tau\right) \tag{2.48a}$$

$$E_z(u,\tau) = \left(\sum_{n=0}^{\infty} e_n(u)\,\tau^n\right) \exp\left(i\,\phi_0(u)/\tau\right) \tag{2.48b}$$

With (2.48a), (2.48b) eqs. (2.47a), (2.47b) represent power series in τ, which vanish identically in τ. All coefficients of the series must then necessarily vanish giving for the first two orders the conditions

$$O(1): \quad \underline{\underline{D}} \cdot \underline{y}_0 = 0 \tag{2.49a}$$

$$O(\tau): \quad \underline{\underline{A}} \cdot \underline{y}_0' + \underline{\underline{B}} \cdot \underline{y}_0 + \underline{\underline{D}} \cdot \underline{y}_1 = 0 \tag{2.49b}$$

with

$$\underline{\underline{A}} = \begin{pmatrix} 2i\phi_0' & 0 \\ 0 & 2i\beta^2\phi_0' \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} i\left(\phi_0'' - \frac{\varepsilon_0'}{\varepsilon_0}\phi_0'\right) & i\phi_0'\alpha_0\beta^2\frac{\varepsilon_0'}{\varepsilon_0} \\ 0 & i\phi_0''\beta^2 \end{pmatrix}$$

$$\underline{\underline{D}} = \begin{pmatrix} -(\phi_0')^2 + \varepsilon - \alpha_0^2 & 0 \\ -\alpha_0(1-\beta^2) & -\beta^2(\phi_0')^2 + \varepsilon_0 \end{pmatrix} \qquad \underline{y}_{0/1} = \begin{pmatrix} b_{0/1} \\ e_{0/1} \end{pmatrix}$$

The linear system (2.49a) is solved with the condition

$$\det\left|D_{ij}\right| = 0$$

by

$$\frac{\phi_0(u)}{\tau} = \pm k_0 \int dz \sqrt{\varepsilon(z) - \alpha_0^2}\,; \quad \underline{y}_0 = C(u)\begin{pmatrix} 1 \\ \frac{\alpha_0}{\varepsilon} \end{pmatrix} \tag{2.50a}$$

and by

$$\frac{\phi_0(u)}{\tau} = \pm k_0 \int dz \sqrt{\frac{\varepsilon(z)}{\beta^2} - \alpha_0^2}\,; \quad \underline{y}_0 = C(u)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.50b}$$

where (2.50a) is called the electromagnetic and (2.50b) the electrostatic solution. The function C(u) is determined by the component of (2.49b) which is orthogonal to $\underline{\underline{D}} \cdot \underline{y}_1$. In the electromagnetic case we find

$$C(u) = C_0 \frac{\sqrt{\varepsilon}}{\sqrt[4]{\varepsilon - \alpha_0^2}} \tag{2.51a}$$

and in the electrostatic case

$$C(u) = C_0 \frac{\sqrt[4]{\varepsilon/\beta^2 - \alpha_0^2}}{\sqrt{\varepsilon/\beta^2}} \tag{2.51b}$$

where C_0 represents an integration constant independent of u.

Summarizing the results we write (2.50), 2.51) in the form

$$k_y^{em,\pm} = \alpha_0 k_0 = k_0\sqrt{\varepsilon}\,\sin\Theta^{em} \tag{2.52a}$$

$$k_z^{em,\pm} = \pm k_0\sqrt{\varepsilon - \alpha_0^2} = \pm k_0\sqrt{\varepsilon}\,\cos\Theta^{em} \tag{2.52b}$$

$$b_0^{em,\pm} = \frac{\sqrt{\varepsilon}}{\sqrt[4]{\varepsilon - \alpha_0^2}}\, C^{em,\pm} \tag{2.52c}$$

$$e_0^{em,\pm} = \frac{\sin\Theta^{em}}{\sqrt[4]{\varepsilon - \alpha_0^2}}\, C^{em,\pm} \tag{2.52d}$$

$$k_y^{es,\pm} = k_0\alpha_0 = k_0\sqrt{\varepsilon/\beta^2}\,\sin\Theta^{es} \tag{2.53a}$$

$$k_z^{es,\pm} = \pm k_0\sqrt{\varepsilon/\beta^2 - \alpha_0^2} = \pm k_0\sqrt{\varepsilon/\beta^2}\,\cos\Theta^{es} \tag{2.53b}$$

$$b_0^{es,\pm} = 0 \tag{2.53c}$$

$$e_0^{es,\pm} = \frac{\cos\Theta^{es}}{\sqrt[4]{\varepsilon/\beta^2 - \alpha_0^2}}\, C^{es,\pm} \tag{2.53d}$$

with integration constants $C^{em,\pm}$, $C^{es,\pm}$. The expressions (2.52), (2.53) correspond to the ray picture of geometrical optics. The refraction laws (2.52a), (2.53a) define the local angles of incidence θ^{em}, θ^{es} enclosed by the wave vectors and the density gradient, while the angle dependence of (2.52d), (2.53d) agrees with the turning of the wave polarization vectors in the plane of incidence. Finally, we remark on a peculiarity of the WKB solution for electrostatic waves. The turning of the polarization vector $\propto \cos\theta^{es}$ in (2.53d) is a result of the first order contribution b_1 to the magnetic field. It was therefore not obtained by Piliya[6], who solved eq. (2.47b) in the electrostatic approximation $B_x = 0$.

2.3.2 Series solutions

Linear first-order systems of ordinary differential equations containing a pole of the first order are conveniently treated by the method of Frobenius[42,43]. The general method, as described in Ref. 44, is introduced and then applied to expand solutions in the vicinity of the electrostatic wave-turning point.

Let us consider the system

$$\underline{w}'(z) = \underline{\underline{A}}(z) \cdot \underline{w}(z) \tag{2.54}$$

where $\underline{\underline{A}}(z)$ is a nxn matrix with a Laurent expansion

$$\underline{\underline{A}}(z) = \frac{1}{z} \sum_{m=0}^{\infty} z^m \underline{\underline{A}}_m \quad ; \quad \underline{\underline{A}}_0 \neq 0 \tag{2.55}$$

convergent on the domain

$$0 < |z| < r \quad ; \quad r > 0 \tag{2.56}$$

To remove the singularity there, we introduce

$$s = \log z \quad ; \quad \underline{v}(s) = \underline{w}(\exp(s)) \tag{2.57}$$

in the half-plane

$$\mathrm{Re}(s) < \log r \tag{2.58}$$

and get from (2.54), (2.55)

$$\underline{v}'(s) = \sum_{m=0}^{\infty} e^{ms} \underline{\underline{A}}_m \cdot \underline{v}(s) \tag{2.59}$$

We now represent the solution by the formal series

$$\underline{y}(s) = \sum_{m=0}^{\infty} e^{ms}\, \underline{y}_m(s) \tag{2.60a}$$

with

$$\underline{y}_m(s) = \underline{p}_m(s)\, e^{\lambda s} \tag{2.60b}$$

where $\underline{p}_m(s)$ are polynomials in s. Inserting (2.60) into (2.59) and equating the coefficients of e^{ms} yields

$$\underline{p}_0'(s) + \{ \lambda \underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_0(s) = 0 \tag{2.61a}$$

$$\underline{p}_m'(s) + \{(\lambda+m)\underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_m(s) = \sum_{j=0}^{m-1} \underline{\underline{A}}_{m-j} \cdot \underline{p}_j(s) \quad ; \quad m \geq 1 \tag{2.61b}$$

To show that (2.61) really determines n independent polynomial solutions, we consider the differential equation

$$\underline{p}' + \underline{\underline{B}} \cdot \underline{p} = \underline{q} \tag{2.62}$$

where $\underline{q}$ is a polynomial of degree d and $\underline{\underline{B}}$ is a constant n x n matrix. As a result of matrix theory to any matrix $\underline{\underline{M}}$ there exists a regular matrix $\underline{\underline{T}}$ so that

$$\underline{\underline{T}}^{-1} \cdot \underline{\underline{M}} \cdot \underline{\underline{T}} = \begin{pmatrix} \underline{\underline{J}}_1 & & 0 \\ & \underline{\underline{J}}_2 & \\ & & \ddots \\ 0 & & \underline{\underline{J}}_l \end{pmatrix} \tag{2.63a}$$

with r_i x r_i matrices

$$\underline{\underline{J}}_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix} \quad ; \quad i \in \{1, 2, \cdots l\} \tag{2.63b}$$

and with the determinant

$$\det \left| \underline{\underline{T}}^{-1} \cdot \underline{\underline{M}} \cdot \underline{\underline{T}} \right| = \prod_{i=1}^{l} (\lambda_i)^{r_i} \; ; \quad n = \sum_{i=1}^{l} r_i \tag{2.63c}$$

Therefore $\underline{\underline{B}}$ can generally be assumed to be of the canonical form (2.63).

If $\underline{\underline{B}}$ is regular, then there exists a unique polynomial solution to (2.62) of degree d. Existence is proved by explicit construction:

$$\underline{p} = \underline{\underline{B}}^{-1} \cdot \underline{q} - \underline{\underline{B}}^{-1} \cdot \underline{\underline{B}}^{-1} \cdot \underline{q}' + \underline{\underline{B}}^{-1} \cdot \underline{\underline{B}}^{-1} \cdot \underline{\underline{B}}^{-1} \cdot \underline{q}'' - + \cdots =$$

$$= \underline{\underline{B}}^{-1} \cdot \underline{\underline{D}}_B \cdot \underline{q} \; ; \quad \underline{\underline{D}}_B = \sum_{n=0}^{d} (-1)^n (\underline{\underline{B}}^{-1})^n \frac{d^n}{ds^n} \tag{2.64}$$

The solution (2.64) is unique since the homogeneous equation ($\underline{q} = 0$) has no polynomial solutions.

If $\underline{\underline{B}}$ is singular, then we may choose

$$\lambda_1 = 0$$

and obtain from (2.63b) the system

$$p_1' = q_1 - p_2$$
$$\vdots$$
$$p_i' = q_i - p_{i+1}$$
$$\vdots$$
$$p_{r_1}' = q_{r_1}$$

which can be successively solved, starting from the last equation. To the homogeneous equation ($q_i = 0$) there exist r_1 polynomial solutions

$$\underline{p}^{j-1} \quad ; \quad j \in \{1, 2, \cdots r_1\}$$

where $\underline{p}^j$ is of degree j. A special solution to the inhomogeneous equation ($q_i \neq 0$) is

$$p_1 = S q_1 - SS q_2 + SSS q_3 - + \cdots - (-1)^{r_1} \underbrace{\int\int\cdots\int}_{r_1 \times} q_{r_1}$$

$$\vdots$$

$$p_{r_1-1} = S q_{r_1-1} - SS q_{r_1}$$

$$\vdots$$

$$p_{r_1} = S q_{r_1}$$

which is of degree $\leq d + r_1$.

From this discussion it follows that for every eigenvalue λ_i to $\underline{\underline{A}}_0$ there exist r_i polynomial solutions to (2.61a) of degree $\leq (r_i - 1)$. For any such solution (2.61b) can be successively solved where the degree of the polynomial $\underline{p}_m$ can only increase if $\lambda_i + m$ is again an eigenvalue to $\underline{\underline{A}}_0$. In this way

$$n = \sum_{i=1}^{\iota} r_i$$

solutions are obtained. Without proof we note that every formal solution is a holomorphic solution in the half-plane (2.58) and all n solutions form a complete fundamental system.

We will now apply this theory to the wave equations of mode conversion (2.30). If $\varepsilon_0(z)$ is an analytic function, then the logarithmic derivative $\varepsilon_0'(z) \; / \varepsilon_0(z)$ has a pole of the first order at the electrostatic wave-turning point ($\varepsilon_0(z) = 0$). In the vicinity of this point we assume

$$\varepsilon_0(z) = - z / L \quad ; \quad L = - 1 / \varepsilon_0'(0) \tag{2.65}$$

Let us consider first the cold case. From (2.31), (2.65) one obtains the expressions

$$\underline{w}(s) = \begin{pmatrix} B_x \\ s B_x' \end{pmatrix} \; ; \quad \underline{\underline{A}}(s) = \frac{1}{s} \begin{pmatrix} 0 & 1 \\ s^2 (s+q) & 2 \end{pmatrix} \tag{2.66a}$$

$$s = - \varepsilon (k_0 L)^{2/3} \; ; \quad q = \alpha_0^2 (k_0 L)^{2/3}$$

for (2.54),

$$\underline{\underline{A}}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} ; \quad \underline{\underline{A}}_2 = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} ; \quad \underline{\underline{A}}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.66b}$$

$$\underline{\underline{A}}_m = 0 \; ; \quad m \notin \{0, 2, 3\}$$

for (2.55) and

$$\begin{aligned} u_0' + \lambda u_0 &= w_0 && m = 0 \\ w_0' + (\lambda - 2) w_0 &= 0 \end{aligned}$$

$$\begin{aligned} u_1' + (\lambda + 1) u_1 &= w_1 && m = 1 \\ w_1' + (\lambda - 1) w_1 &= 0 \end{aligned} \tag{2.66c}$$

$$\begin{aligned} u_2' + (\lambda + 2) u_2 &= w_2 && m = 2 \\ w_2' + \lambda w_2 &= q u_0 \end{aligned}$$

$$\begin{aligned} u_m' + (\lambda + m) u_m &= w_m && m \geq 3 \\ w_m' + (\lambda + m - 2) w_m &= q u_{m-2} + u_{m-3} \end{aligned}$$

with

$$\underline{p}_m = \begin{pmatrix} u_m \\ w_m \end{pmatrix}$$

for (2.61). The eigenvalues of the matrix $\underline{\underline{A}}_0$ are

$$\lambda_1 = 2 \quad ; \quad \lambda_2 = 0$$

For λ_1 the solution to (2.66c) is readily found since all polynomials

are of zeroth degree:

$$u_0 = c \quad ; \quad u_1 = 0 \quad ; \quad u_2 = \frac{q}{8} c$$

$$u_m = \frac{1}{m} \frac{1}{m+2} \left(q u_{m-2} + u_{m-3} \right) \quad ; \quad m \geq 3$$

For λ_2 we use (2.64) and obtain from (2.66c)

$$u_0 = c \; ; \quad u_1 = 0 \; ; \quad u_2 = \frac{q}{2} s c \tag{2.67}$$

$$u_m = \frac{1}{m} \frac{1}{m-2} D_m D_{m-2} \left(q u_{m-2} + u_{m-3} \right) ; \quad m \geq 3$$

The differential equation (2.67) is solved with a polynomial ansatz

$$u_m = \frac{q}{2} c \, a_{m-2} s + b_m \quad ; \quad m \geq 0$$

yielding

$$a_{-2} = a_{-1} = 0 \quad ; \quad a_0 = 1$$

$$a_m = \frac{1}{m} \frac{1}{m+2} \left(q a_{m-2} + a_{m-3} \right) \quad m \geq 1$$

and

$$b_0 = c \quad ; \quad b_1 = b_2 = 0$$

$$b_m = \frac{1}{m} \frac{1}{m-2} \left(q b_{m-2} + b_{m-3} - q(m-1) a_{m-2} \right) ; \quad m \geq 3$$

The general solution to (2.31) with the profile (2.65) is then

$$B_x = C_2 \left(C_1 B_1 + B_2 \right) \tag{2.68}$$

$$B_1 = P_1(s) \; ; \quad B_2 = P_2(s) + \frac{q}{2} P_1(s) \log s$$

$$P_1(s) = \sum_{m=0}^{\infty} \alpha_m s^{m+2} \; ; \quad P_2(s) = \sum_{m=0}^{\infty} \beta_m s^m$$

with the recurrence relations

$$\alpha_0 = 1 \; ; \; \alpha_1 = 0 \; ; \; \alpha_2 = \frac{q}{8} \; ; \; \alpha_m = \frac{\alpha_{m-3} + q \alpha_{m-2}}{m(m+2)} \; ; \; m \geq 3$$

$$\beta_0 = 1 \; ; \; \beta_1 = \beta_2 = 0 \; ; \; \beta_m = \frac{q \beta_{m-2} + \beta_{m-3} - (m-1) q \alpha_{m-2}}{m(m-2)}$$

and constants C_1, C_2. This solution is well known from previous works[17,19,45] but no derivation has been given there.

As a simple application we discuss the behaviour of the electromagnetic field near the singularity. From (2.28), (2.68) and with the complex dielectric constant

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} + i \frac{\omega_p^2}{\omega^2} \frac{\nu}{\omega} \tag{2.69}$$

one obtains in the limit $s \to 0$

$$B_x = C_2 \left\{ 1 + C_1 s^2 + \frac{q}{2} s^2 \log s + \cdots \right\} \tag{2.70a}$$

$$E_z = -q C_2 \left\{ s^{-1} + C_1 s + \frac{q}{2} s \log s + \cdots \right\} \tag{2.70b}$$

$$E_y = -i (k_0 L)^{1/3} C_2 \left\{ 2 C_1 + \frac{q}{2} + q \log s + \cdots \right\} \tag{2.70c}$$

Accordingly, the magnetic field is a continuous function. If it does not vanish for $s = 0$, then E_z has a pole of the first order, E_y diverges logarithmically and the time-averaged Poynting vector

$$\langle S_z \rangle = \frac{c}{8\pi} \mathrm{Re}\left\{ - E_y^* B_x \right\} = \frac{c}{8\pi} |C_2|^2 (k_0 L)^{1/3} (-q\varphi + 2\,\mathrm{Im}\, C_1^*)$$

$$\varrho = |\varrho| \exp\{i\varphi\}$$

jumps discontinuously by the amount

$$[\langle S_z \rangle] = - \frac{c}{8} q (k_0 L)^{1/3} |C_2|^2 \tag{2.71}$$

The thermal case is now treated in an analogous way. From (2.30), (2.65) one gets

$$\underline{w}(\xi) = \begin{pmatrix} H \\ \xi H' \\ G \\ \xi G' \end{pmatrix}; \quad \underline{\underline{A}}(\xi) = \xi^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \beta^2 \xi^3 & 2 & -p(1-\beta^2)\xi^3 & 0 \\ 0 & 0 & 0 & 1 \\ \xi^2 & 0 & \xi^2(\xi+p) & 1 \end{pmatrix}$$

with

$$p = \beta^{4/3}(1-\beta^2) q; \quad \xi = \left(\frac{k_0 L}{\beta}\right)^{2/3} \frac{z}{L}; \quad E_z = \alpha_0 \left(\frac{k_0 L}{\beta}\right)^{2/3} (1-\beta^2)\, G$$

The polynomial equations (2.61) then assume the form

$$\underline{p}_0' + \{ \lambda \underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_0 = 0 \tag{2.72a}$$

$$\underline{p}_1' + \{ (\lambda+1) \underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_1 = 0 \tag{2.72b}$$

$$\underline{p}_2' + \{ (\lambda+2) \underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_2 = \underline{\underline{A}}_2 \cdot \underline{p}_0 \quad , \quad m \geq 3 \tag{2.72c}$$

$$\underline{p}_m' + \{ (\lambda+m) \underline{\underline{I}} - \underline{\underline{A}}_0 \} \cdot \underline{p}_m = \underline{\underline{A}}_3 \cdot \underline{p}_{m-3} + \underline{\underline{A}}_2 \cdot \underline{p}_{m-2} \tag{2.72d}$$

where

$$\underline{\underline{A}}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \underline{\underline{A}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & p & 0 \end{pmatrix}; \quad \underline{\underline{A}}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta^2 & 0 & -p(1-\beta^2) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\underline{\underline{A}}_m = 0 \quad ; \quad m \notin \{0, 2, 3\}$$

For the eigenvalues λ of $\underline{\underline{A}}_0$ the systems (2.72a) - (2.72c) are easily solved, yielding

$$\underline{\lambda = 0}: \quad \underline{p}_0 = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad \underline{p}_1 = 0$$

$$\underline{p}_2 = C_1 \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 0 \\ P/2 \\ P \end{pmatrix}$$

$$\underline{\lambda = 1}: \quad \underline{p}_0 = C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \; ; \quad \underline{p}_1 = 0 \; ; \quad \underline{p}_2 = C_3 \begin{pmatrix} 0 \\ 0 \\ P/6 \\ P/2 \end{pmatrix}$$

$$\underline{\lambda = 2}: \quad \underline{p}_0 = C_4 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \; ; \quad \underline{p}_1 = 0 \; ; \quad \underline{p}_2 = C_4 \begin{pmatrix} 0 \\ 0 \\ 1/12 \\ 1/3 \end{pmatrix}$$

with constants C_1, C_2, C_3, C_4. The matrix

$$(\lambda + m)\, \underline{\underline{I}} - \underline{\underline{A}}_0 \quad ; \quad m \geq 3$$

is regular and therefore all polynomials are of zeroth degree. Then (2.72d) is an algebraic equation with the solution

$$x_m = \frac{1}{\lambda+m} \frac{1}{\lambda+m-2} \left\{ \beta^2 x_{m-3} - \frac{P}{1-\beta^2} y_{m-3} \right\}$$

$$u_m = \frac{1}{\lambda+m-2} \left\{ \beta^2 x_{m-3} - \frac{P}{1-\beta^2} y_{m-3} \right\}$$

$$y_m = \frac{1}{\lambda+m} \frac{1}{\lambda+m-1} \left\{ y_{m-3} + x_{m-2} + P\, y_{m-2} \right\}$$

$$v_m = \frac{1}{\lambda+m-1} \left\{ y_{m-3} + x_{m-2} + P\, y_{m-2} \right\}$$

$$\underline{p}_m = \begin{pmatrix} x_m \\ u_m \\ y_m \\ v_m \end{pmatrix} .$$

It can be summarized in the form

$$\begin{pmatrix} \Gamma \\ G \end{pmatrix} = \sum_{m=0}^{\infty} \underline{z}_m(\lambda)\, \xi^{m+\lambda} \tag{2.73}$$

with

$$\underline{z}_0(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \underline{z}_0(1) = C_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \underline{z}_0(2) = C_4 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underline{z}_1(\lambda) = 0$$

$$\underline{z}_2(0) = C_1 \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ P/2 \end{pmatrix} ; \quad \underline{z}_2(1) = C_3 \begin{pmatrix} 0 \\ P/6 \end{pmatrix} ; \quad \underline{z}_2(2) = C_4 \begin{pmatrix} 0 \\ 1/12 \end{pmatrix}$$

$$\underline{z}_m(\lambda) = \underline{\underline{U}}_1 \cdot \underline{z}_{m-3}(\lambda) + \underline{\underline{U}}_2 \cdot \underline{z}_{m-2}(\lambda) \quad ; \quad m \geq 3$$

$$\underline{\underline{U}}_1 = \frac{1}{\lambda+m} \begin{pmatrix} \frac{\beta^2}{\lambda+m-2} & \frac{-P/(1-\beta^2)}{\lambda+m-2} \\ 0 & \frac{1}{\lambda+m-1} \end{pmatrix}$$

$$\underline{\underline{U}}_2 = \frac{1}{\lambda+m} \begin{pmatrix} 0 & 0 \\ \frac{1}{\lambda+m-1} & \frac{P}{\lambda+m-1} \end{pmatrix}$$

Using (2.28), (2.73) and the definition

$$w(0) = \left(\frac{\beta}{k_0 L}\right)^{2/3} \left(\alpha_0\, H(0)\right)^{-1} E_z(0) \tag{2.74}$$

the integration constants may be expressed as

$$C_1 = H(0) \; ; \quad C_2 = H(0)\,\frac{w(0)}{1-\beta^2}$$

$$C_3 = H(0)\,\frac{w'(0)}{1-\beta^2} \; ; \quad C_4 = \frac{i}{2}\,\beta^{4/3}\,(k_0L)^{-1/3}\,E_y(0) \tag{2.75}$$

The first orders of the expansion then assume the more usual form[6]:

$$H(s) = H(0)\left\{1 + \frac{1}{3}\left(1 - \frac{1-\beta^2}{\beta^{2/3}}\,q\,w(0)\right)s^3\right\} +$$

$$+ \frac{i}{2}\,(k_0L)^{-1/3}\,E_y(0)\,s^2 + O(s^4)$$

$$E_z(\xi) = \alpha_0\left(\frac{k_0L}{\beta}\right)^{2/3}(1-\beta^2)\;H(0)\left\{\frac{1}{2}\,\xi^2 + \frac{w(0)}{1-\beta^2}\left(1 + \frac{p}{2}\,\xi^2 + \frac{1}{6}\,\xi^3\right) +\right.$$

$$\left. + \frac{w'(0)}{1-\beta^2}\left(\xi + \frac{p}{6}\,\xi^3\right) + O(\xi^4)\right\}$$

In contrast to the cold plasma solution (2.70) in (2.73) neither poles nor logarithms are present.

2.3.3 Solution in the resonance layer

An alternative way to solve the thermal problem near the singularity is to rescale the wave equations. The method is well known from boundary layer problems in hydrodynamics[46]. In this solution boundary conditions are conveniently regarded.

Regular perturbation techniques are not applicable to (2.30), where the

small parameter $\beta^2 \ll 1$ is a coefficient of the highest derivative. Within a temperature-dependent resonance width around the singular point electrostatic modes are of essential importance. Within this layer the variables are assumed to scale according to the cold solution (2.70) as

$$s = \delta\,\xi \; ; \quad E_z = \alpha_0\,(k_0L)^{2/3}\,\frac{a}{\delta} \tag{2.76}$$

$$E_y = b + c\,\log\delta \; ; \quad H = B_x - q\beta^2\,\frac{a}{\delta}$$

where ξ, a, b, c and B_x represent finite quantities for $\delta(\beta) \longrightarrow 0$. With (2.30), (2.65) and (2.76) one obtains

$$\frac{1}{\delta^2}\left(H'' - \frac{1}{\xi}\,H'\right) - \delta\,\xi H = -q\,\xi\,a$$

$$\frac{\beta^2}{\delta^3}\,a'' - \left(\xi + q\,\frac{\beta^2}{\delta}\right)a = H$$

In the limit $\delta \longrightarrow 0$ the second derivative of the electric field must be a leading term. This condition determines the scaling factor to

$$\delta = \beta^{2/3} \tag{2.77}$$

and yields the rescaled wave equations

$$H'' - \frac{1}{\xi}\,H' = 0 \tag{2.78a}$$

$$a'' - \xi\,a = H \tag{2.78b}$$

The solution to (2.78a) in the domain $\mathbb{C} \setminus \{0\}$ is

$$H = H(0) + \frac{i}{2}\,\beta^{4/3}\,(k_0L)^{-1/3}\,E_y(0)\,\xi^2 \longrightarrow B_x(0)$$

and therefore (2.78b) can be rewritten in the common form of the in-

homogeneous Airy equation

$$w'' - \xi w = 1 \tag{2.79}$$

with

$$a(\xi) = \mathcal{B}_x(0)\, w(\xi)$$

Its general solution is[47]

$$w = \pi \left\{ Hi + C_1 (Ai + i\, Bi) + C_2 (Ai - i\, Bi) \right\}$$

The Airy functions

$$Ai = \pi^{-1} \int_0^\infty dt \cos\left(\tfrac{1}{3}t^3 + \xi t\right) \tag{2.80a}$$

$$\mathcal{B}i = \pi^{-1} \int_0^\infty dt \sin\left(\tfrac{1}{3}t^3 + \xi t\right) + \pi^{-1} \int_0^\infty dt \exp\left(-\tfrac{1}{3}t^3 + \xi t\right)$$

are the solutions to the homogeneous Airy equation and π H_i with

$$Hi = \pi^{-1} \int_0^\infty dt \exp\left(-\tfrac{1}{3}t^3 + \xi t\right) \tag{2.80b}$$

$$Hi(0) = \tfrac{2}{3}\, \mathcal{B}i(0) \; ; \quad Hi'(0) = \tfrac{2}{3}\, \mathcal{B}i'(0)$$

is a particular solution to the inhomogeneous equation. For $\xi \to \pm \infty$ the asymptotic representations are

$$(Ai \pm i\, Bi)(-\xi) \longrightarrow \pi^{-1/2} (-\xi)^{-1/4} \exp\left\{ \mp i\, \tfrac{2}{3} (-\xi)^{3/2} \pm i\, \tfrac{\pi}{4} \right\}$$

$$(-\xi) \gg 1$$

$$Hi(-\xi) \longrightarrow \pi^{-1}\, \xi^{-1} \tag{2.81}$$

$$Ai(\xi) \longrightarrow \tfrac{1}{2}\, \pi^{-1/2}\, \xi^{-1/4} \exp\left\{ -\tfrac{2}{3}\, \xi^{3/2} \right\}$$

$$(\xi \gg 1) \tag{2.81}$$

$$\mathcal{B}i(\xi),\, Hi(\xi) \longrightarrow \pi^{-1/2}\, \xi^{-1/4} \exp\left\{ \tfrac{2}{3}\, \xi^{3/2} \right\}$$

The boundary conditions that no wave is incident for $(-\xi) \gg 1$ and that the field decreases for $\xi \gg 1$ then require

$$C_1 = 0 \; ; \quad C_2 = -i \tag{2.82}$$

yielding the solution[20]

$$w(\xi) = \pi \left\{ Hi - i\, (Ai - i\, Bi) \right\} \tag{2.83}$$

From (2.80), (2.83) the integral representation[6]

$$w(\xi) = (-i) \int_0^\infty dt \exp\left(-i\, t^3/3 - i\, \xi t \right) \tag{2.84}$$

follows. The asymptotic solutions are

$$w(\xi) = -\tfrac{i}{2}\, \pi^{1/2}\, \xi^{-1/4} \exp\left(-\tfrac{2}{3}\, \xi^{3/2} \right) \; ; \quad \xi \gg 1 \tag{2.85}$$

$$w(\xi) = \xi^{-1} - \pi^{1/2} (-\xi)^{-1/4} \exp\left\{ i\, \tfrac{2}{3} (-\xi)^{3/2} + i\, \tfrac{\pi}{4} \right\} \; ; \quad (-\xi) \gg 1$$

Figure 2 shows a numerical evaluation of the integral representation (2.84). The amplitude $/w/^2$ has a maximum of about 3.5 and a half-width of about 1.6. In contrast to the cold solution the wave maximum ($-\xi \approx 1.9$) is shifted from the singular point ($\xi = 0$).

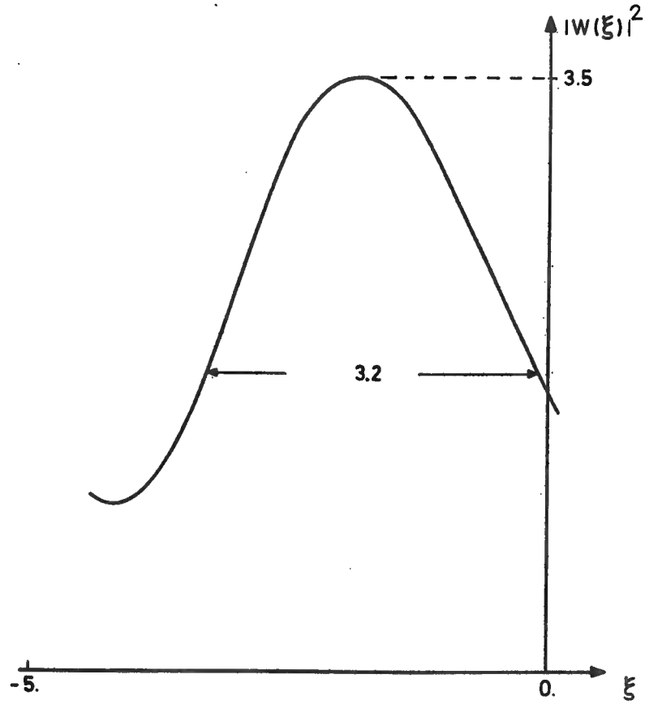


Fig. 2: Numerical evaluation of the integral representation (2.83) of $/w/^2$. $w(\xi)$ is the solution to the inhomogeneous Airy equation (2.79) satisfying the boundary conditions (2.82).

2.4 Boundary value problem

The numerical integration of the wave equations (2.30) is conveniently done for an initial value problem with initial values given in the overdense region. We shall now establish the general boundary value problem of linear mode conversion and then reduce it to an initial value problem.

To formulate the boundary conditions, we rewrite (2.30) in the form of a first-order system

$$\underline{Y}'(z) = \underline{\underline{A}}(z) \cdot \underline{Y}(z) \tag{2.86}$$

and assume slow variation of the dielectric constant at the boundaries of the integration interval

$$z_1 \leq z \leq z_2$$

with

$$\varepsilon(z_1) - \alpha_0^2 > 0 \quad ; \quad \varepsilon(z_2) < 0$$

We then define two fundamental sets of solutions to (2.86)

$$\underline{\underline{U}}(z) = (\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4)$$

$$\left.\begin{aligned} \underline{u}_1 &= \underline{e}_1 \exp\{-\phi^{em}\} \\ \underline{u}_2 &= \underline{e}_2 \exp\{-\phi^{es}\} \\ \underline{u}_3 &= \underline{e}_3 \exp\{+\phi^{em}\} \\ \underline{u}_4 &= \underline{e}_4 \exp\{+\phi^{es}\} \end{aligned}\right\} \quad z \to z_2$$

and

$$\underline{\underline{W}}(z) = (\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4)$$

$$\left.\begin{aligned} \underline{w}_1 &= \underline{f}_1 \exp\{i\,\phi_+^{em}\} \\ \underline{w}_2 &= \underline{f}_2 \exp\{i\,\phi_+^{es}\} \\ \underline{w}_3 &= \underline{w}_1^* \quad ; \quad \underline{w}_4 = \underline{w}_2^* \end{aligned}\right\} \quad z \to z_1$$

with usual WKB amplitudes $\underline{e}_i$ (z), $\underline{f}_i$ (z) and phases

$$\phi_\pm^{em} = \int dz\, (\pm\varepsilon \mp \alpha_0^2)^{1/2} \quad ; \quad \phi_\pm^{es} = \int dz\, \left(\pm\frac{\varepsilon}{\beta^2} \mp \alpha_0^2\right)^{1/2}$$

The boundary value problem of linear mode conversion consists in determining a solution

$$\underline{\underline{Y}}(z) = \underline{\underline{U}}(z) \cdot \underline{a} = \underline{\underline{W}}(z) \cdot \underline{c}$$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \quad ; \quad \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{2.87a}$$

to (2.86) with given constants

$$c_1 \neq 0 \quad ; \quad c_2 = a_3 = a_4 = 0 \tag{2.87b}$$

The four unknown expansion coefficients a_1, a_2, c_3, c_4 are determined from the system

$$\underline{c} = \underline{\underline{L}} \cdot \underline{a} \quad ; \quad \underline{\underline{L}} = \underline{\underline{W}}^{-1} \cdot \underline{\underline{U}} \tag{2.88}$$

where $\underline{\underline{L}}$ is a regular constant matrix. Introducing the real fundamental set of solutions

$$\underline{\underline{V}}(z) = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$$

$$\underline{v}_1 = \mathrm{Re}(\underline{w}_1) \; ; \quad \underline{v}_2 = \mathrm{Jm}(\underline{w}_1)$$

$$\underline{v}_3 = \mathrm{Re}(\underline{w}_2) \; ; \quad \underline{v}_4 = \mathrm{Jm}(\underline{w}_2)$$

the matrix $\underline{\underline{L}}$ is given by

$$\underline{\underline{L}} = \underline{\underline{N}} \cdot \underline{\underline{M}} \quad ; \quad \underline{\underline{M}} = \underline{\underline{V}}^{-1} \cdot \underline{\underline{U}} \tag{2.89}$$

$$\underline{\underline{N}} = \underline{\underline{W}}^{-1} \cdot \underline{\underline{V}} = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & +i & 0 & 0 \\ 0 & 0 & 1 & +i \end{pmatrix}$$

and $\underline{\underline{M}}$ represents a real matrix. It can be determined at the boundary z_1 where $\underline{\underline{V}}$ (z_1) is known if $\underline{\underline{U}}(z_1)$ is numerically calculated, starting with the initial condition $\underline{\underline{U}}(z_2)$. From (2.87b), (2.88), (2.89) the expansion coefficients and the reflection coefficient R are given by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underline{\underline{P}}^{-1} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \; ; \quad \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \underline{\underline{P}}^{*} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$R = \left| c_3/c_1 \right|^2 = \left| \left(\underline{\underline{P}}^{*} \cdot \underline{\underline{P}}^{-1} \right)_{11} \right|^2$$

with

$$\underline{\underline{P}} = \frac{1}{2} \begin{pmatrix} M_{11} - i\,M_{21} & M_{12} - i\,M_{22} \\ M_{31} - i\,M_{41} & M_{32} - i\,M_{42} \end{pmatrix}$$

2.5 Energy conservation theorem

The fluid model (2.20) is consistent with an energy conservation law. Its local and global forms are derived. We shall use the notation

$$y_i(\underline{x},t) = \mathrm{Re}\left\{ y_i(\underline{x})\, e^{-i\omega t} \right\}$$

$$\langle y_i(\underline{x},t)\, y_j(\underline{x},t) \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} dt\; y_i(\underline{x},t)\, y_j(\underline{x},t) = \tag{2.90}$$

$$= \frac{1}{2}\, \mathrm{Re}\left\{ y_i(\underline{x})\, y_j(\underline{x})^* \right\}$$

and note that

$$\left(D\, y(\underline{x})\right)^* = D\left(\mathrm{Re}\, y(\underline{x})\right) - i\, D\left(\mathrm{Im}\, y(\underline{x})\right) = D\left(y(\underline{x})\right)^*$$

for any linear differential operator D with real coefficients.

As a general consequence of Maxwell's equations the law of energy conservations for the electromagnetic field is established by Poynting's theorem:

$$\nabla \cdot \langle \underline{S}(\underline{x},t) \rangle = - \langle \underline{\underline{E}}(\underline{x},t) \cdot \underline{j}(\underline{x},t) \rangle \tag{2.91a}$$

with the Poynting vector

$$\underline{S}(\underline{x},t) = \frac{c}{4\pi}\, \underline{\underline{E}}(\underline{x},t) \times \underline{B}(\underline{x},t) \tag{2.91b}$$

We shall now show that the r.h.s. of (2.91a) is equivalent to the divergence of a current density. In terms of the fluid model (2.20) one has

$$\underline{j}(\underline{x},t) = \mathrm{Re}\left\{ \underline{j}(\underline{x})\, e^{-i\omega t} \right\} \tag{2.92}$$

$$\underline{j}(\underline{x}) = \frac{i}{4\pi\omega} \left\{ \omega_p^2\, \underline{\underline{E}}(\underline{x}) - S^2\, \nabla(\nabla \cdot \underline{\underline{E}}(\underline{x})) \right\}$$

and therefore

$$\langle \underline{\underline{E}}(\underline{x},t) \cdot \underline{j}(\underline{x},t) \rangle = \frac{1}{4} \left(\underline{\underline{E}}(\underline{x}) \cdot \underline{j}^*(\underline{x}) + c.c. \right) =$$

$$= \frac{S^2}{16\pi\omega} \left(i\, \underline{\underline{E}}(\underline{x}) \cdot \nabla(\nabla \cdot \underline{\underline{E}}^*(\underline{x})) + c.c. \right) =$$

$$= \frac{S^2}{16\pi\omega} \left(i\, \nabla \cdot \left\{ \underline{\underline{E}}(\underline{x})\, \nabla \cdot \underline{\underline{E}}^*(\underline{x}) \right\} + c.c. \right) \tag{2.93}$$

Then from (2.91), (2.93) we find

$$\nabla \cdot \left\{ \langle \underline{S}(\underline{x},t) \rangle + \langle \underline{\underline{T}}(\underline{x},t) \rangle \right\} = 0 \tag{2.94a}$$

with

$$\langle \underline{S}(\underline{x},t) \rangle = \frac{c}{16\pi} \left\{ \underline{\underline{E}}(\underline{x}) \times \underline{B}^*(\underline{x}) + c.c. \right\} \tag{2.94b}$$

and

$$\langle \underline{\underline{T}}(\underline{x},t) \rangle = \frac{c}{16\pi} \frac{\beta^2}{k_0} \left\{ i\, \underline{\underline{E}}(\underline{x})\, \nabla \cdot \underline{\underline{E}}^*(\underline{x}) + c.c. \right\} \tag{2.94c}$$

We proceed by evaluating (2.94) for electromagnetic and electrostatic waves. A single electromagnetic wave

$$k_0 \underline{B} = \underline{k} \times \underline{E} \;;\quad \underline{k} \cdot \underline{\underline{E}} = 0 \;;\quad k^2 = k_0^2\, \varepsilon$$

contributes only to the Poynting vector:

$$\langle \underline{S} \rangle = \frac{c_{gr}}{8\pi} \left| \underline{\underline{E}}(\underline{x}) \right|^2 \frac{\underline{k}}{k} \;;\quad \langle \underline{\underline{T}} \rangle = 0 \;;\quad c_{gr} = c\sqrt{\varepsilon} \tag{2.95a}$$

while a single electrostatic wave

$$\underline{B} = 0 \quad ; \quad k^2 = k_0^2 \frac{\varepsilon}{\beta^2} \quad ; \quad \underline{k} \parallel \underline{E}$$

gives only a thermal contribution

$$\langle \underline{T} \rangle = \frac{S_{gr}}{8\pi} |\underline{E}(x)|^2 \frac{\underline{k}}{k} \quad ; \quad \langle \underline{S} \rangle = 0 \quad ; \quad S_{gr} = S\sqrt{\varepsilon} \qquad (2.95b)$$

In both cases the energy current density is just the group velocity times the energy density of the mode[30,42]. If there are several waves in the medium then in addition to the current densities (2.95) there arise cross terms

$$\langle \underline{S}_{ij} \rangle = \frac{c}{16\pi} \{ \underline{E}_i \times \underline{B}_j^* + c.c. \}$$

$$\langle \underline{T}_{ij} \rangle = \frac{c}{16\pi} \frac{\beta^2}{k_0} \{ i \underline{E}_i \nabla \cdot \underline{E}_j^* + c.c. \}$$

between mode i and mode j. To show that these contributions vanish along a particular surface normal direction $\underline{n}$, we consider the cases:

(1) Two electromagnetic waves:

Here one has

$$\langle \underline{T}_{ij} \rangle = 0$$

while

$$\underline{n} \cdot \langle \underline{S}_{ij} \rangle = 0$$

leads to

$$\underline{k}_i^{em} \cdot \{ \hat{\underline{B}}_i \times (\hat{\underline{B}}_j \times \underline{n}) \} + \underline{k}_j^{em} \cdot \{ \hat{\underline{B}}_j \times (\hat{\underline{B}}_i \times \underline{n}) \} = 0$$

with unit vectors $\hat{\underline{B}}_{i/j}$ along the directions of $\underline{B}_{i,j}$. In the special

case of p-polarization

$$\hat{\underline{B}}_i = \hat{\underline{B}}_j \quad ; \quad \hat{\underline{B}}_{i/j} \cdot \underline{n} = 0$$

one readily gets

$$(\underline{k}_i^{em} + \underline{k}_j^{em}) \cdot \underline{n} = 0 \qquad (2.96a)$$

(2) Two electrostatic waves

Here one has

$$\langle \underline{S}_{ij} \rangle = 0$$

and

$$\underline{n} \cdot \langle \underline{T}_{ij} \rangle = 0$$

gives

$$(\underline{k}_i^{es} + \underline{k}_j^{es}) \cdot \underline{n} = 0$$

(3) One electromagnetic, one electrostatic wave:

Here the condition

$$\underline{n} \cdot \{ \langle \underline{T}_{ij} \rangle + \langle \underline{S}_{ij} \rangle \} = 0$$

is rewritten as

$$(\underline{k}^{es} - \underline{k}^{em}) \cdot \hat{\underline{B}} \times \underline{n} = 0 \qquad (2.96c)$$

As an application consider the conversion of a p-polarized electromag-

netic wave (o) into a reflected one (1) and an electrostatic wave. Here the wave vectors are

$$(k_z^{em})^o = -(k_z^{em})^1 \; ; \; (k_y^{em})^o = (k_y^{em})^1 = k_y^{es}$$

and according to (2.96) all cross terms vanish in the z direction:

$$\langle S_z \rangle + \langle T_z \rangle = \langle S_z^{em} \rangle^o + \langle S_z^{em} \rangle^1 + \langle T_z^{es} \rangle \qquad (2.97)$$

For this case a global energy law is easily derived. Integrating (2.94) with the boundary conditions

$$\langle S_z \rangle \Big|_{z=+\infty} = 0 \; ; \; \langle T_z \rangle \Big|_{z=+\infty} = 0$$

yields

$$\langle S_z^{em} \rangle^o + \langle S_z^{em} \rangle^1 + \langle T_z^{es} \rangle = 0$$

or equivalent by

$$A + R = 1$$

with

$$A = -\frac{\langle T_z^{es} \rangle}{\langle S_z^{em} \rangle^o} \; ; \; R = -\frac{\langle S_z^{em} \rangle^1}{\langle S_z^{em} \rangle^o} \qquad (2.98)$$

Using (2.85), (2.95) one finds

$$\langle S_z^{em} \rangle^o = (1-\alpha_o^2)^{1/2} \frac{c}{8\pi} |C_o|^2$$

$$\langle T_z^{es} \rangle = -\frac{c}{8} k_o L \alpha_o^2 |B_x(0)|^2$$

and with (2.98)

$$A = \pi k_o L \frac{\alpha_o^2}{(1-\alpha_o^2)^{1/2}} \left| \frac{B_x(0)}{C_o} \right|^2 \qquad (2.99)$$

where $|C_o|$ is the electromagnetic vacuum amplitude.

We finally derive (2.99) from the cold plasma equations. Inserting the current density

$$\underline{j}(\underline{x}) = \frac{i\omega}{4\pi} \frac{\omega_p^2}{\omega^2} \left(1 + i\frac{\nu}{\omega}\right)^{-1} \underline{E}(\underline{x}) \qquad (2.100)$$

into (2.91) gives in the limit $\frac{\nu}{\omega} \to 0$:

$$\nabla \cdot \langle \underline{S} \rangle = -\frac{\nu}{8\pi} \frac{\omega_p^2}{\omega^2} |\underline{E}|^2$$

Assuming (2.27) integration with the boundary condition

$$\langle S_z \rangle \Big|_{z=+\infty} = 0$$

yields

$$\langle S_z \rangle = \lim_{\nu/\omega \to 0} \int_z^{+\infty} dz' \; \frac{\nu}{\omega} \frac{\omega_p^2}{\omega^2} |\underline{E}|^2$$

For a further evaluation of the integral the domain of integration may be split into three intervals $(-\infty, -a)$, $(-a, a)$, $(a, +\infty)$, where a is chosen within the region $(\nu/\omega) k_o L \ll k_o a \ll (k_o L)^{1/3}$, $L = -1/\varepsilon'(0)$. Then, in the collisionless limit, only the intermediate interval covering the resonance region $|z| \ll \frac{\nu}{\omega} L$ gives a finite contribution. On the other hand, since $k_o z \ll (k_o L)^{1/3}$ there, the expressions (2.70) for the fields are valid, leading to

$$\langle S_z \rangle = \lim_{\nu/\omega \to 0} \int_{z/L}^{+\infty} du \; \frac{\alpha_o^2 L |B_x|^2 \omega_p^2}{8\omega} \left(\frac{1}{\pi} \frac{\nu/\omega}{u^2 + (\nu/\omega)^2} \right)$$

The resonance function in the integrant is a representation of the

δ -distribution[48] and we therefore find

$$\langle S_z \rangle = \frac{c}{8\pi} k_0 L \, \alpha_0^2 \, |B_x(0)|^2 \, \theta(-z) \tag{2.101}$$

with

$$\theta(z) = \begin{cases} 1 & ; \; z > 0 \\ 0 & ; \; z < 0 \end{cases}$$

Defining

$$A = \frac{\langle S_z \rangle}{\langle S_z^{em} \rangle^0} \tag{2.102}$$

one again obtains (2.99). We note that (2.101) agrees with the jump condition (2.71) for the Poynting vector.

3.1 Linear mode conversion for smooth layers

Numerical results for linear density profiles are represented and compared with analytic solutions. We shall discuss the field envelopes, the time-dependent fields and the absorption curve. Good agreement is found for temperatures of present laser plasmas ($\beta^2 \approx 0.01$).

The complete set of wave equations (2.30) supplemented by the boundary conditions (2.87b) was solved for the parameters

$$\varepsilon(z) = -z/L \; ; \quad S = k_0 z / (k_0 L)^{1/3} \; ; \quad q = \alpha_0^2 (k_0 L)^{2/3}$$

$$\beta^2 = 0.01 \; ; \quad k_0 L = 10^3 \; ; \quad |C_0| = 1$$

where $|C_0|$ is the amplitude of the incident electromagnetic wave. We first discuss the field structure as given by the squared magnitudes $|B_x|^2$ and $|E_z|^2$ of the field components (Figs. 3 and 4). This representation is conveniently interpreted as describing:

(1) the maximum amplitudes in every plane z = const:

$$y^2(z,t) = |y(z)|^2 \cos^2(\varphi(z) + k_0 \alpha_0 y - \omega t) \leq |y(z)|^2$$

$$y(z) = |y(z)| \, e^{i\varphi(z)}$$

(2) the time-averaged distribution:

$$\langle y^2(z,t) \rangle = \frac{1}{2} |y(z)|^2$$

(3) the envelope of the time-dependent fields[+)]

[+)] This interpretation was given by P. Mulser (private communication)

The equation

$$F(u, z, t) = u - |y(z)|^2 \cos^2(\varphi(z) + k_0 \alpha_0 y - \omega t) = 0$$

gives a family of curves u = u(z, t) in the (u, z) plane with parameter t. From

$$\partial_t F(u, z, t) = 2\omega \cos(\varphi + k_0 \alpha_0 y - \omega t) \sin(\varphi + k_0 \alpha_0 y - \omega t) = 0$$

the envelope equations u = 0 and $u = |y(z)|^2$ are found.

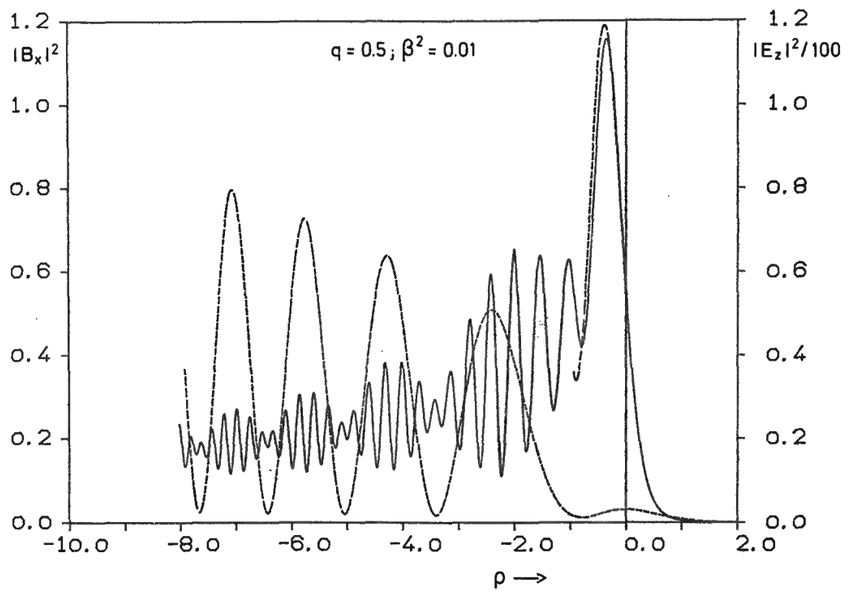


Fig. 3: Field structure in an inhomogeneous plasma layer as a result of electromagnetic wave reflection and plasma wave excitation. The magnetic field amplitude $|B_x|^2$ ---- and the electric field amplitude $|E_z|^2$ ——— are shown as functions of $\zeta = k_0 z / (k_0 L)^{1/3}$, which increases towards higher plasma densities. At the critical density, $\zeta = 0$, the numerical solution is compared with the resonance layer approximation (2.84) of the electric field -----.

The fields are easily analyzed in the WKB domain

$$\zeta << -(1 + q)$$

Using (2.52) one obtains for the magnetic field

$$B_x = \{ \exp(i\phi_0) + r_1 \exp(i\phi_1) \} |B_0| \exp(i\psi) \qquad (3.1)$$

$$|B_0|^2 = \frac{\sqrt{-\zeta}}{(k_0 L)^{1/3}} \left(\frac{1 - q/(k_0 L)^{2/3}}{1 - q/(-\zeta)} \right)^{1/2} |C_0|^2 \; ; \quad r_1 = \sqrt{R}$$

$$\phi_{0/1}(\zeta) = \pm \frac{2}{3} \{ (-\zeta_m - q)^{3/2} - (-\zeta - q)^{3/2} \}$$

where ζ_m is the position of a maximum of $|B_x|^2$, and ψ a constant phase. Taking the magnitude of (3.1) yields

$$|B_x|^2 = \{ 1 + r_1^2 + 2 r_1 \cos(\phi_0 - \phi_1) \} |B_0|^2 \qquad (3.2)$$

From (3.2) it is seen that the local amplitude $|B_0|^2$ is modulated by half an electromagnetic wavelength, decreases in proportion to $\sqrt{-\zeta}$ and scales at the critical density with $(k_0 L)^{-1/3}$. The amplitude of the maxima and minima are found to be

$$|B_x|^2_{max} = (1 + r_1)^2 |B_0|^2 \; ; \; |B_x|^2_{min} = (1 - r_1)^2 |B_0|^2$$

It is noted that in the case of maximum absorption (Fig. 3) the wave is still strongly modulated:

$$R = 0.5 \; ; \; (1 - r_1)^2 \approx 0.09 \; ; \; (1 + r_1)^2 \approx 2.91$$

The wavelength near the turning point is estimated from

$$(\phi_0 - \phi_1) \Big|_{\zeta_0}^{\zeta_1} = 2\pi \; ; \quad \zeta_1 = -(1 + q)$$

to be

$$\lambda = (k_0 L)^{1/3} (\xi_1 - \xi_0) \; ; \qquad \xi_1 - \xi_0 = \left(\tfrac{3}{2}\pi + 1\right)^{2/3} - 1 \approx 2.2 \tag{3.3}$$

It increases with the density and scales with $(k_0 L)^{1/3}$.

Analogously one finds from (2.52), (2.53), (2.99) for the electric field

$$E_z = \left\{ \exp(i\phi_0) + r_1 \exp(i\phi_1) + r_2 \exp(i\phi_2) \right\} |E_0| e^{i\Psi} \tag{3.4}$$

with

$$|E_0|^2 = \frac{q}{-\xi} \frac{(k_0 L)^{1/3}}{\sqrt{-\xi}} \left(\frac{1 - q/(k_0 L)^{2/3}}{1 - q/(-\xi)} \right)^{1/2} |C_0|^2$$

$$\phi_2(\xi) = \frac{2}{3} \left(-\xi - \beta^{4/3} q \right)^{3/2} + \tilde{\Psi} \; ; \qquad \xi = \beta^{-2/3} \xi$$

$$r_2 = \left\{ \frac{A}{\beta} \frac{-\xi}{q} \left(1 - q/(-\xi)\right)^{1/2} \left(1 - \beta^2 q/(-\xi)\right)^{1/2} \right\}^{1/2}$$

where $\tilde{\Psi}$ is again a constant phase. The squared magnitude of (3.4) yields

$$|E_z|^2 = \left\{ 1 + r_1^2 + r_2^2 + 2 r_1 \cos(\phi_1 - \phi_0) + \right.$$

$$\left. + 2 r_2 \left[\cos(\phi_2 - \phi_0) + r_1 \cos(\phi_2 - \phi_1) \right] \right\} |E_0|^2 \tag{3.5}$$

The local amplitude of the incident wave $/E_0/^2$ contains the polarization factor $q/(-\xi)$ and the scalar amplitude $(k_0 L)^{1/3} / \sqrt{-\xi}$ with the scaling proportional to $(k_0 L)^{1/3}$. The modulation in Fig. 3 is essentially caused by the terms proprotional to r_2 ($r_2 \gg r_1$). Since the difference in their wave numbers is small

$$\frac{d}{dz} \phi_2 \approx \frac{1}{\beta} \frac{d}{dz} \phi_1 = 10 \frac{d}{dz} \phi_1 = -10 \frac{d}{dz} \phi_0$$

a characteristic beat-wave picture is obtained:

$$\cos(\phi_2 - \phi_0) + r_1 \cos(\phi_2 - \phi_1) =$$

$$(1 + r_1) \cos\left(\frac{\phi_0 - \phi_1}{2}\right) \cos\left(\phi_2 - \frac{\phi_0 + \phi_1}{2}\right) + (1 - r_1) \sin\left(\frac{\phi_0 - \phi_1}{2}\right) \sin\left(\phi_2 - \frac{\phi_0 + \phi_1}{2}\right)$$

$|E_z|^2 \longrightarrow$

$$\left\{ r_2^2 + 2(1 + r_1) r_2 \cos\left(\frac{\phi_0 - \phi_1}{2}\right) \cos\left(\phi_2 - \frac{\phi_0 + \phi_1}{2}\right) \right\} |E_0|^2 \tag{3.6}$$

$$r_2 \gg 1 \; ; \qquad \frac{1 - r_1}{1 + r_1} \ll 1$$

It is noted that the position of the nodes in the beat-wave pattern (3.6)

$$(\phi_0 - \phi_1)/2 = \frac{2n+1}{2} \pi \qquad ; \qquad n = 0, 1, \ldots$$

is identical with the position of the minima in the magnetic field (3.2).

In the resonance layer ($/\xi/ \lesssim 1$; $/\xi/ \lesssim 0.2$) we find from (2.99)

$$\left| \frac{B_x(0)}{C_0} \right|^2 = (1 - \alpha_0^2)^{1/2} (k_0 L)^{-1/3} \frac{A}{\pi q} \approx 0.03$$

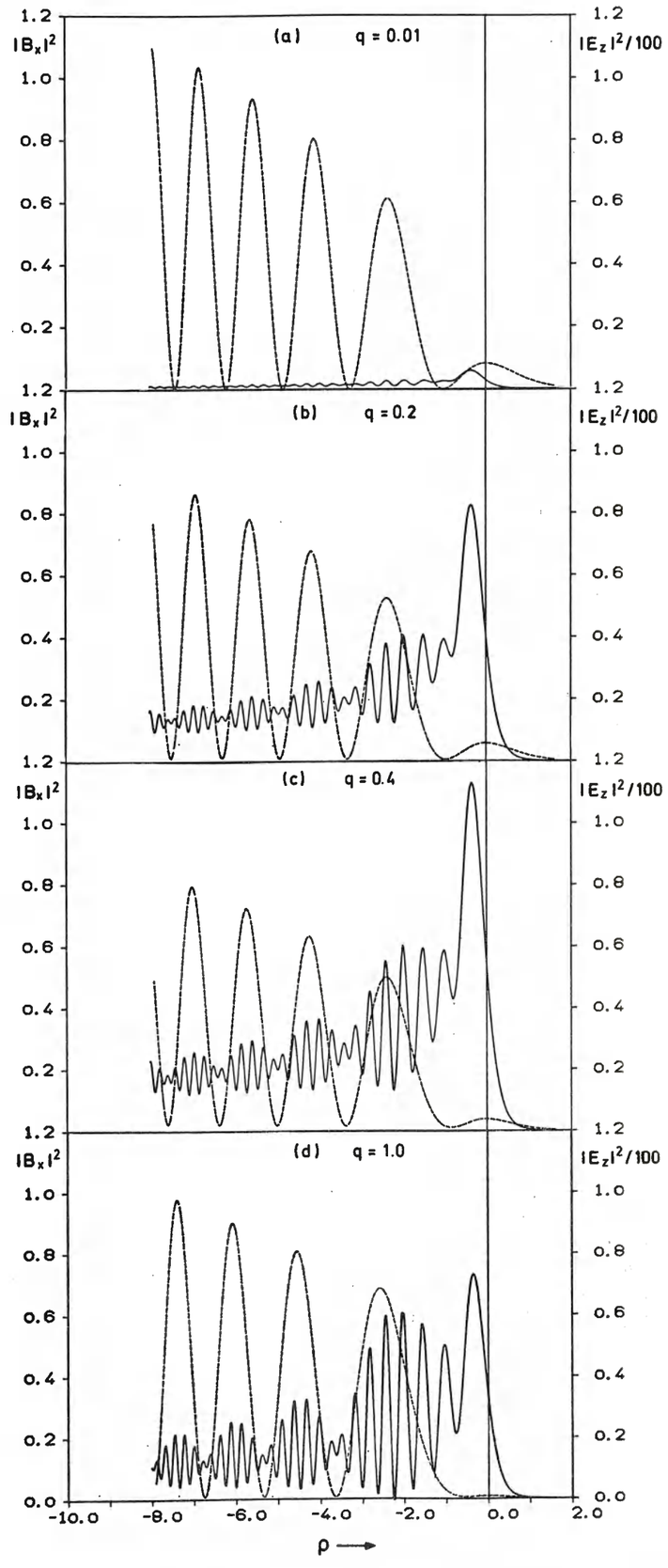
and using (2.76), (2.84)

$$\left| \frac{E_z}{C_0} \right|^2 = (1 - \alpha_0^2)^{1/2} (k_0 L)^{+1/3} \frac{A}{\pi} \frac{|W|^2}{\beta^{4/3}} \tag{3.7}$$

In Fig. 3 the resonance field (3.7) is compared with the numerical solution. The agreement is excellent near the critical density and qualitatively good up to the first electrostatic wave minimum.

Figure 4 shows field structures for q values which are different from the optimum angle for resonance absorption. At low q values, $q \ll 0.5$, the absorption coefficient varies linearly with q^{18} . In this case $r_2 \gg 1$ becomes independent of q, and thus for arbitrarily low q values qualitatively the same beat modulation as in Fig. 3 is obtained (Fig. 4a, b, c). However, since the incoming wave amplitude $/E_o/^2$ is proportional to q the total amplitude (3.6) decreases (Fig. 4a). The situation is different for large q values, $q \gg 0.5$. Here r_2 decreases with increasing q, and for $r_2/r_1 \ll 1$ the modulation is exclusively due to the standing light wave (Fig. 4d, e, f). In this case the turning point recedes from the critical surface and no effective wave coupling is possible (Fig. 4f).

The time-dependent aspects of the fields are seen from Fig. 5 representing the real fields $E_z(z, t)$, $B_x(z, t)$ for the same parameters as in Fig. 3. The single pictures differ in phase by $\pi/6$ and describe half an oscillation period between two zeros of the magnetic field at the critical density in (a) and (g). The outgoing electrostatic wave is clearly recognized. At the critical density the magnetic field and the electric field show a phase difference of about $\pi/2$ in agreement with the resonant excitation of the plasma wave.



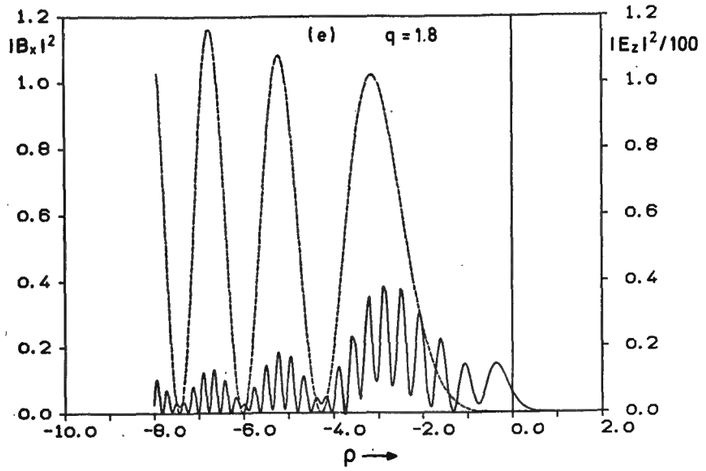


Fig. 4: Field structures as a function of the angle parameter $q = (L k_0)^{2/3} \alpha_0^2$ At small angles of incidence, $q \ll 0.5$, the electric field component E_z parallel to the density gradient is small but strongly modulated by the electrostatic wave (a). The resonance field reaches maximum values at $q \approx 0.5$. For large angles $q \gg 0.5$ the wave turning points $\zeta = -q$ and $\zeta = -\beta^2 q$ of the electromagnetic and electrostatic waves separate and no coupling occurs. In this case a purely standing electromagnetic wave results (f).

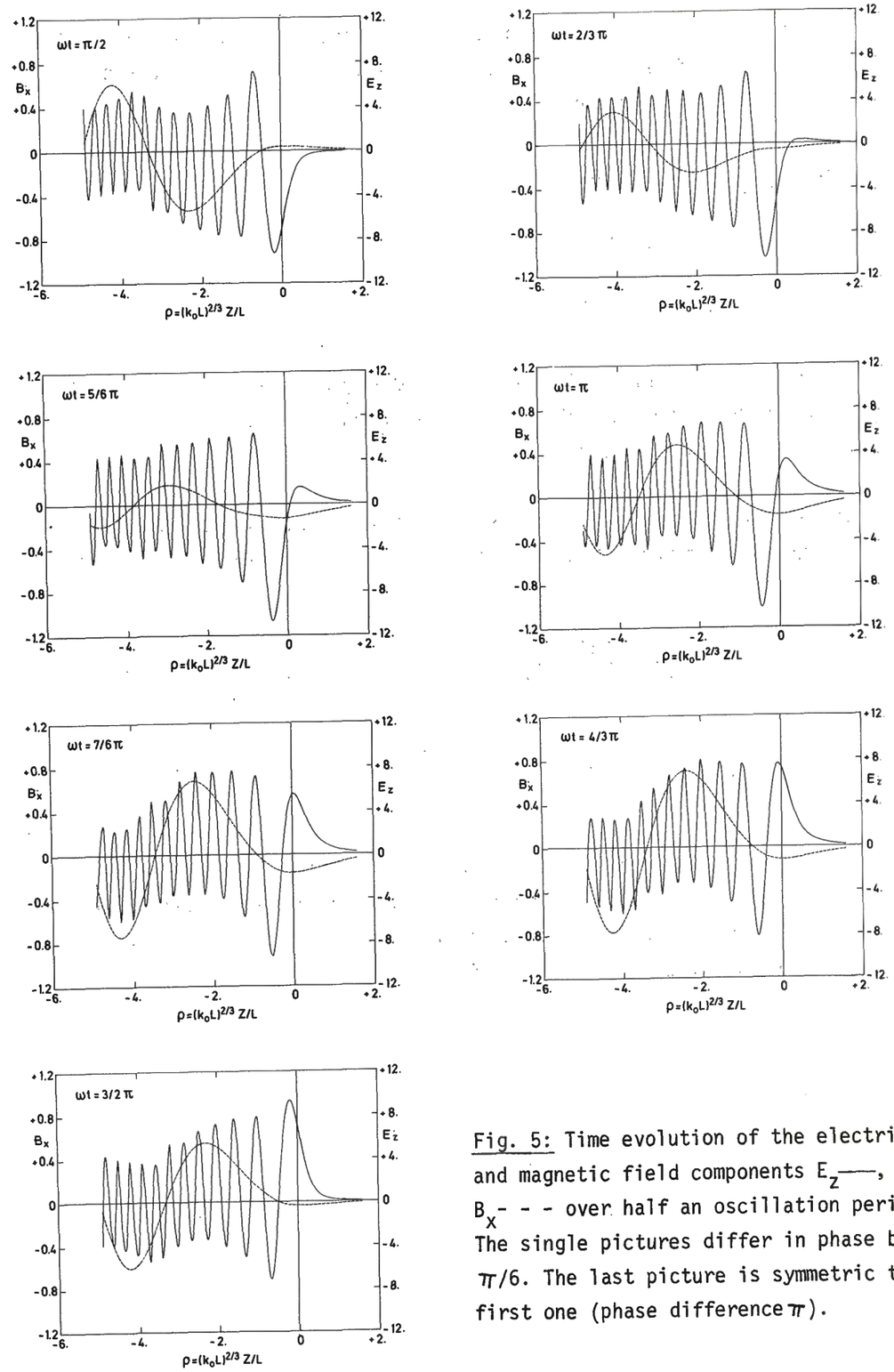


Fig. 5: Time evolution of the electric and magnetic field components E_z——, B_x- - - over half an oscillation period. The single pictures differ in phase by $\pi/6$. The last picture is symmetric to the first one (phase difference π).

Let us now consider the absorption coefficient. The cold-plasma solution for linear profiles is easily proved to depend only on the parameter q. Imposing the boundary condition

$$B_x \Big|_{\xi = +\infty} = 0$$

on the solution (2.68) yields

$$B_x = c_2 \, f(\xi, q)$$

where C_2 is a normalization constant and $f(\xi, q)$ a function of ξ and q only. By definition the absorption coefficient is independent of C_2 and ξ and hence at most a function of q.

The absorption curve A = A(q) is well known from previous work[18,19]. It reaches a maximum of about 0.5 at $q \approx 0.5$ and decreases to zero according to

$$A = \begin{cases} \left\{ \dfrac{2\pi}{3^{1/3}\,\Gamma(1/3)} \right\}^2 q & ; \quad q << 1 \\ 2 \exp\left\{ -\dfrac{4}{3} q^{3/2} \right\} & ; \quad q >> 1 \end{cases}$$

Our numerical results for $\beta^2 = 0.01$ agree well with the cold-plasma solution (Fig. 6(a)). At high temperatures ($\beta^2 \gtrsim 0.1$) we observe an increase in the absorption maximum and a shift to higher q values (Fig. 6(b)). This behaviour was also found in Ref. 5.

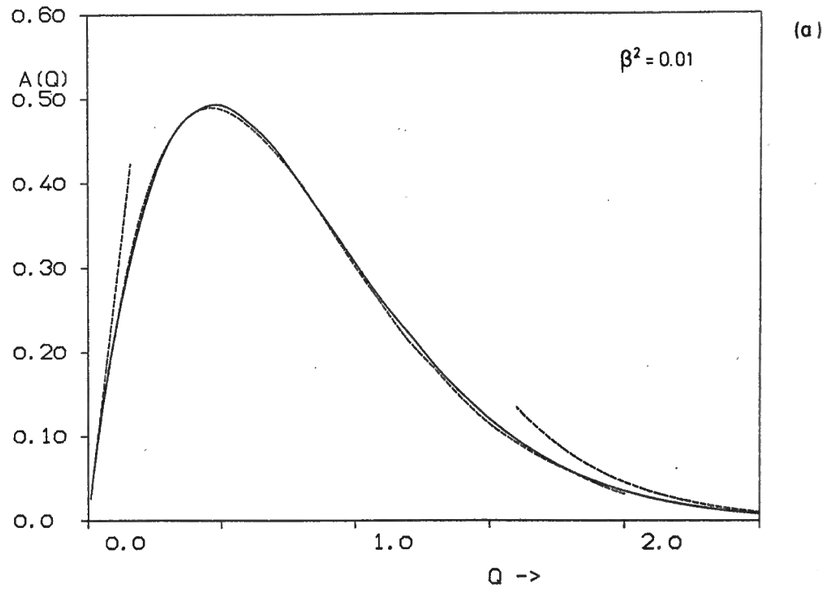


Fig. 6: Resonance absorption at smooth density gradients as a function of the angle and temperature parameters $q = (k_0 L)^{2/3} \sin^2 \theta_0$ and $\beta^2 = 3\,T/mc^2$ respectively. The numerical solution agrees well with the analytic solutions taken from Ref. 18, 19 up to temperatures of about 20 keV (a). At higher temperatures deviations from the cold-plasma approximation occur (b).

3.2 Transition to thin layers ($\beta^2 = 0$)

Precise limitations of the linear profile model can be established by connecting the nonuniform layer to regions of constant densities. This model will be used to discuss a continuous transition from smooth to thin layers in a cold plasma.

We now assume the dielectric constant to be of the form

$$\varepsilon(z) = \begin{cases} \varepsilon_1 > 0 & ; \quad z \in (-\infty, -\varepsilon_1 L) \\ -z/L & ; \quad z \in (-\varepsilon_1 L, -\varepsilon_2 L) \\ \varepsilon_2 < 0 & ; \quad z \in (-\varepsilon_2 L, +\infty) \end{cases} \tag{3.8}$$

with solutions to (2.31) given by

$$B_x(z) = \begin{cases} C_3 B_3 + C_4 B_4 & ; \quad z \in (-\infty, -\varepsilon_1 L) \\ C_2 (C_1 B_1 + B_2) & ; \quad z \in (-\varepsilon_1 L, -\varepsilon_2 L) \\ C_0 B_0 & ; \quad z \in (-\varepsilon_2 L, +\infty) \end{cases} \tag{3.9a}$$

where B_1, B_2 are the linear profile solutions (2.68) and

$$B_0 = \exp\{-k_0 \sqrt{-\varepsilon_2 + \alpha_0^2}\, z\}$$

$$B_{3/4} = \exp\{\pm i k_0 \sqrt{\varepsilon_1 - \alpha_0^2}\, z\} \tag{3.9b}$$

the WKB solutions (2.52). To relate the amplitudes of (3.9), we integrate (2.31) over continuous boundary layers located at z_1 and z_2:

$$\left[\log\left(\frac{B_x'}{\varepsilon}\right)\right] + k_0^2 \int_\Delta dz\, (\varepsilon - \alpha_0^2)\, \frac{B_x}{B_x'} = 0 \tag{3.10}$$

In the limit $\Delta \to 0$ one gets from (2.31)

$$[B_x] = 0 \tag{3.11a}$$

and from (3.10)

$$\left[\frac{B_x'}{\varepsilon}\right] = 0 \tag{3.11b}$$

Using now (3.9a), (3.11), one easily derives

$$R = \left|\frac{C_4}{C_3}\right|^2 = \left|\frac{\Delta_{01}(z_2)\Delta_{32}(z_1) - \Delta_{02}(z_2)\Delta_{31}(z_1)}{\Delta_{01}(z_2)\Delta_{42}(z_1) - \Delta_{02}(z_2)\Delta_{41}(z_1)}\right|^2 \tag{3.12}$$

with

$$\Delta_{ij}(z) = B_i(z) B_j'(z) - B_i'(z) B_j(z)$$

and inserting (2.68), (3.9b) into (3.12) yields

$$R = \left|\frac{C^-}{C^+}\right|^2 \tag{3.13}$$

with

$$\mathrm{Re}(C^\mp) = \{P_2'(2) + m_2 P_2(2)\}\{P_1'(1) + \frac{q}{2} P_2(1)(\frac{1}{s_1} \mp \pi m_1)\} -$$

$$- P_2'(1)\{m_2 P_1(2) + P_1'(2) + \frac{q}{2}\frac{P_2(2)}{s_2}\} +$$

$$+ \frac{q}{2} P_2'(1) \ln\left|\frac{s_1}{s_2}\right| \{P_2'(2) + m_2 P_2(2)\}$$

$$Im(C^{\mp}) = \mp \{ P_2'(2) + m_2 P_2(2) \} \{ m_1 P_1(1) \pm \tfrac{q}{2} \mp P_2'(1) \} \pm$$

$$\pm m_1 P_2(1) \{ m_2 P_1(2) + P_1'(2) + \tfrac{q}{2} \frac{P_2(2)}{S_2} \} \mp$$

$$\mp \tfrac{q}{2} m_1 P_2(1) \ln \left| \frac{S_1}{S_2} \right| \{ P_2'(2) + m_2 P_2(2) \}$$

$$m_i = \sqrt{\mp S_i \mp q} \quad ; \quad S_i = -(k_0 L)^{2/3} \varepsilon_i \quad ; \quad i \in \{1,2\}$$

$$P_i(j) = P_i(S_j) \quad ; \quad P_i'(j) = P_i'(S_j) \quad ; \quad j \in \{1,2\}$$

Finally, from (3.13) the absorption coefficient

$$A = 1 - R = 4\pi q m_1 |S_1| \left(P_2'(2) + m_2 P_2(2) \right)^2 / |C^+|^2 \qquad (3.14)$$

follows. In this form it represents an expansion in the coordinates $S_{1/2}$ of the layer boundaries. We shall call the layer thin (smooth) if

$$\min \{ |S_1|, S_2 \} \ll (\gg) \ 1 \qquad (3.15)$$

since in this case the extent $/z_i/$ of the transition layer is much smaller (larger) than the local wavelength:

$$|S_i|^{3/2} = k_0 L |\varepsilon_i|^{3/2} = k_i |z_i| \ ; \quad k_i = k_0 \sqrt{|\varepsilon_i|} \qquad (3.16)$$

The absorption coefficient (3.14) was evaluated for both symmetric ($\tau = |S|_1 / S_2 = 1$) and asymmetric ($\tau \ll 1$, $\tau \gg 1$) layers (Fig. 7). It is seen that the well-known smooth-layer solution is approached for $/S_{1/2}/ \gtrsim 1$, independently of the layer symmetry. The transition to thin layers shows a decrease in absorption which advances more slowly for $\tau \ll 1$ than for $\tau = 1$ and $\tau \gg 1$.

In order to be more explicit, we now consider (3.14) in the limit $/S_{1/2}/ \ll 1$:

$$A = \frac{4\pi |S_1|^{3/2}}{1+\tau} \frac{\alpha_1^2 (1-\alpha_1^2)^{1/2}}{1 - \alpha_1^2 (1-\tau)} \quad ; \quad \alpha_1^2 = 9/|S_1| \qquad (3.17)$$

Here A can at most reach the order of unity if the denominator becomes small. Equating therefore the denominator to zero yields the common surface wave dispersion relation[33]

$$\alpha_1^2 = \frac{1}{1-\tau} \quad <=> \quad \alpha_0^2 = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

In the present model ($\alpha_1^2 < 1$) this condition can only be satisfied in the limit $\tau \to 0$. In this case we obtain from (3.17)

$$A = 4\pi |S_1|^{3/2} \frac{\alpha_1^2 \sqrt{1-\alpha_1^2}}{1+\tau-\alpha_1^2}$$

with the maximum value

$$A_{max} = 2\pi |S_2|^{3/2} \tau \longrightarrow 0$$

at the angle

$$\alpha_1^2 = 1 - \tau \longrightarrow 1$$

From this discussion it can be concluded that absorption decreases and ultimately reaches zero for all angles $\alpha_0^2 < \varepsilon_1$.

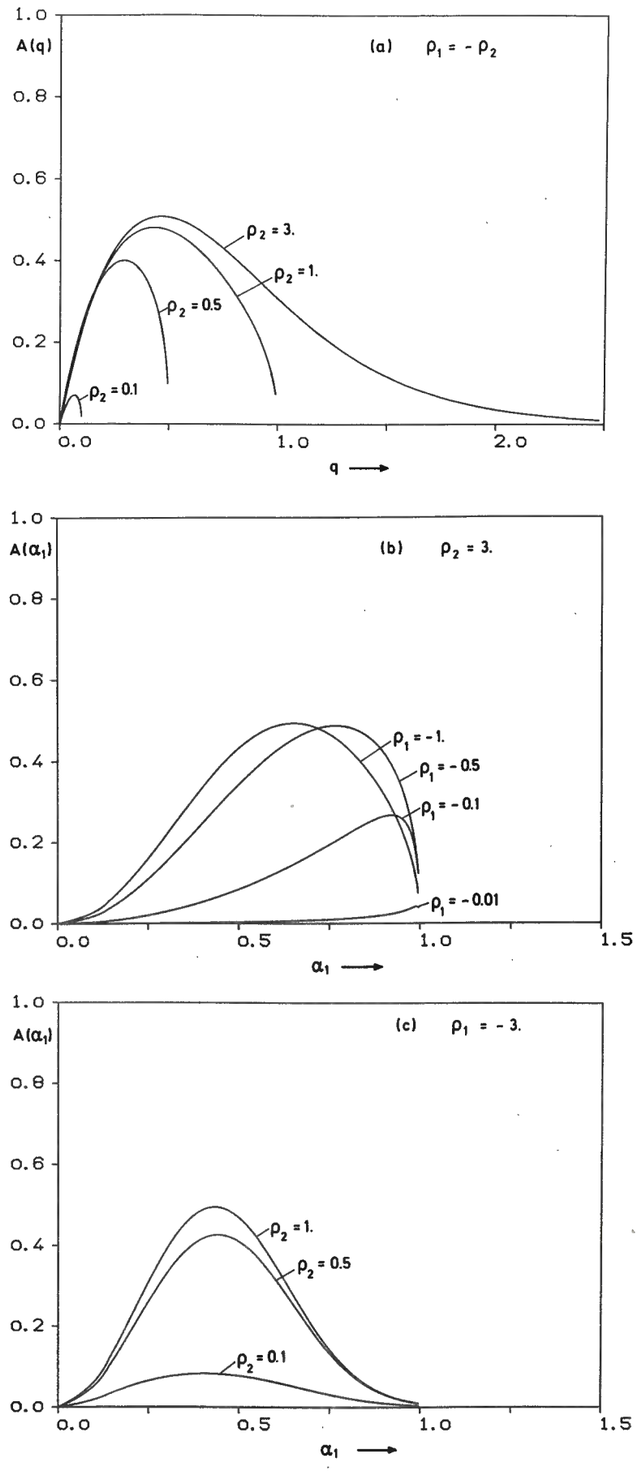


Fig. 7: Absorption curves illustrating the transition from smooth to thin layers in a cold plasma. δ_1, δ_2 are the coordinates of the layer boundaries. In the symmetric case, $\delta_1 = -\delta_2$, absorption is shown versus the Ginzburg parameter q (a). Steepening the profile, $|\delta_{1/2}| \to 0$, resonance absorption is seen to disappear. The same occurs in the asymmetric cases (b), (c), where absorption is shown as a function of the angle of incidence $\alpha_1 = \alpha_0 / \sqrt{\varepsilon_1}$.

Let us now consider the thin-layer approximation more generally. Assuming $|\delta_{1/2}| \ll 1$, we obtain from (2.70), (2.71) the jump conditions

$$[B_x] = c_2 \left\{ c_1 [\delta^2] + \frac{q}{2} [\delta^2 \log \delta] \right\}$$

$$\left[\frac{B_x'}{\varepsilon}\right] = k_0 (k_0 L)^{1/3} q \, c_2 \, [\log \delta]$$

$$[\langle S_z \rangle] = -\frac{c}{8} q (k_0 L)^{1/3} |c_2|^2$$

If the fields are finite at the layer boundaries, then in the limit $L \to 0$

$$c_2 = O(1) \quad ; \quad c_1 = O(L^{-1/3})$$

and one obtains

$$[B_x] = \left[\frac{B_x'}{\varepsilon}\right] = [\langle S_z \rangle] = 0$$

This is the usual Fresnel result[2]. It excludes absorption in the transition layer independently of the density profiles in the underdense and overdense plasma regions. Finally, it is noted that we are here not concerned with intermediate scale lengths and angles $\alpha_0^2 > \varepsilon_1$. In this case strong energy dissipation by surface waves has been described[21].

3.3 Transition to thin layers ($\beta^2 \neq 0$)

In order to discuss thin layers in thermal plasmas, we solve the wave equations for an Epstein transition layer. The thin layer asymptote to the absorption curve is then described by the model of a step profile.

We now consider the dielectric constant

$$\varepsilon(z) = \varepsilon_1 - \frac{\varepsilon_1 - \varepsilon_2}{1 + \exp(-z/\Delta)} \tag{3.16a}$$

with the properties

$$\varepsilon(z_0) = 0 \quad ; \quad z_0 = \Delta \ln \tau \tag{3.16b}$$

$$\varepsilon(z) - \varepsilon(0) = -\{\varepsilon(-z) - \varepsilon(0)\} \tag{3.16c}$$

$$\varepsilon(z) \longrightarrow \begin{cases} \varepsilon_1 & ; \quad z/\Delta << -1 \\ -\frac{z - z_0}{L} & ; \quad \left|\frac{z - z_0}{\Delta}\right| << 1 \\ \varepsilon_2 & ; \quad z/\Delta >> 1 \end{cases} \tag{3.16d}$$

where

$$\alpha_0^2 < \varepsilon_1 \leq 1 \quad ; \quad \varepsilon_2 < 0 \quad ; \quad \Delta > 0$$

$$\tau = \varepsilon_1/(-\varepsilon_2) \quad ; \quad \varepsilon(0) = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2)$$

$$L = -1/\varepsilon'(0) = \frac{\Delta}{\bar{\varepsilon}} \quad ; \quad \frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon_1} + \frac{1}{(-\varepsilon_2)}$$

To establish a condition for the smooth-layer solution to be valid, we require that the regions

$$|z - z_0|/\Delta << 1$$

and

$$s + q << (-1) \quad ; \quad s = k_0(z - z_0)/(k_0 L)^{1/3}$$

overlap. This readily yields the condition

$$(\bar{k}\Delta)^{2/3} >> 1 + q \quad ; \quad \bar{k} = k_0\sqrt{\bar{\varepsilon}} \tag{3.17}$$

which is strongest for large q values. We note that (3.17) is essentially identical with the estimate (3.15), (3.16).

One can also easily give a definition of thin layers in the thermal case. According to the series expansion (2.73) an asymptotic solution is expected for

$$|\xi_{1/2}| << 1 \tag{3.18}$$

where

$$|\xi_i|^{3/2} = \frac{k_0 L}{\beta} |\varepsilon_i|^{3/2} = |z_i| \, k_i^{es}$$

$$k_i^{es} = \frac{k_0}{\beta} \sqrt{|\varepsilon_i|} \quad ; \quad |z_i| = L |\varepsilon_i|$$

compares the layer width with the electrostatic wavelength. With assumption (2.33b) this limit can only be reached for small step heights ($|\varepsilon_i| \to 0$) within our model.

The parameters involved in the problem are now briefly described. The basic equations (2.30), (3.16) contain the constants

$$k_0 \, , \; \varepsilon_1 \, , \; \varepsilon_2 \, , \; \Delta \, , \; \alpha_0^2 \, , \; \beta^2$$

If we substitute

$$\tilde{z} = k_0 \sqrt{\varepsilon_1} \, z \quad ; \quad \tilde{E}_z = \sqrt{\varepsilon_1} \, E_z \quad ; \quad \alpha_1^2 = \frac{\alpha_0^2}{\varepsilon_1}$$

$$\tau = \varepsilon_1 / (-\varepsilon_2) \quad ; \quad \tilde{\Delta} = k_0 \sqrt{\varepsilon_1} \, \Delta = \bar{k} \Delta \, (1+\tau)^{1/2}$$

the explicit dependence on ε_1 and k_0 can be eliminated. The absorption coefficient is therefore generally described by just four parameters:

$$\beta^2 \, , \; \alpha_1^2 \, , \; \tau \, , \; \bar{k} \Delta$$

Figure 8 shows absorption curves as calculated from (2.30), (3.16) for fixed values $\tau = 1$ and $\beta^2 = 0.1$. Deviations from the smooth layer solution are seen for $\bar{k}\Delta \lesssim 1$ (a). As a consequence of the finite layer width these curves have an upper limitation at the angle of grazing incidence

$$\alpha_0 = \sqrt{\varepsilon_1} \quad \Rightarrow \quad q = (1+\tau)(\bar{k}\Delta)^{2/3}$$

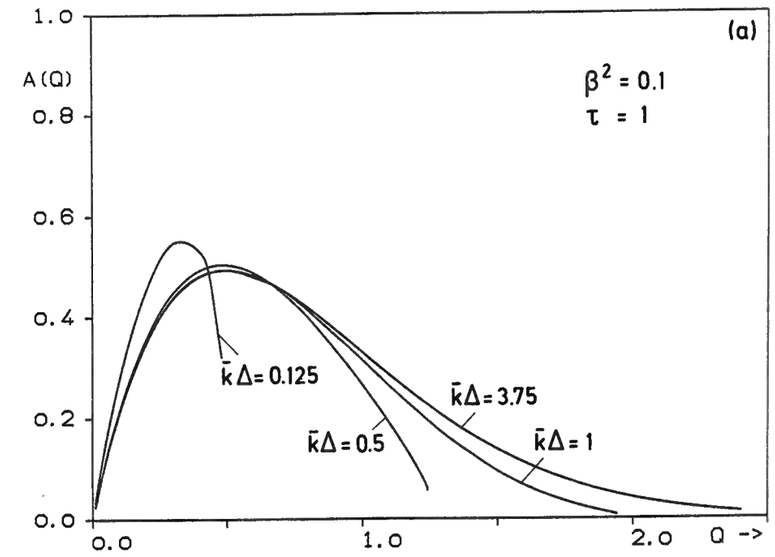


Fig. 8: Light absorption from an Epstein transition layer with variable layer thickness $\bar{k}\Delta$ for a temperature $\beta^2 = 0.1$. For smooth profiles, $\bar{k}\Delta \gtrsim 1$, the Ginzburg resonance curve A = A(q) is seen to be valid (a). With decreasing layer width the absorption maximum shifts to larger angles and for $\bar{k}\Delta \lesssim 0.1$ an asymptote independent of the layer thickness is approached (b). The profile is chosen symmetric to the critical density, $\tau = 1$.

With decreasing values $\bar{k}\Delta$ the absorption maximum is seen to be shifted to larger angles and the whole solution approaches an asymptote independent of $\bar{k}\Delta$ (b).

To explain the thin-layer solution, we consider a configuration of two semi-infinite homogeneous plasma regions 1 ($-\infty < z < 0$) and 2 ($0 < z < +\infty$). We first derive the boundary conditions at the interface z = o. Integrating (2.30) over a continuous transition layer there yields

$$\left[\log\left(\frac{H'}{\varepsilon_0}\right)\right] + k_0^2 \int_\Delta dz\, \varepsilon_0\left(\frac{H}{H'} - \alpha_0 \frac{E_z}{H'}\right) = 0$$

$$\left[E_z'\right] + \left(k_0^2/\beta^2\right) \int_\Delta dz \left\{\left(\varepsilon_0 - \alpha_0^2\beta^2\right) E_z - \alpha_0 H\right\} = 0 \qquad (3.19)$$

In the limit $\Delta \to 0$ the second-order equations (2.30) require

$$[H] = 0 \quad ; \quad [E_z] = 0$$

and from (3.19) one obtains

$$\left[\frac{H'}{\varepsilon_0}\right] = 0 \quad ; \quad \left[E_z'\right] = 0$$

Finally, using (2.28) we get the result

$$[B_x] = [E_y] = [E_z] = [E_z'] = 0 \qquad (3.20)$$

It is noted that

$$E_z = E_z^{em} + E_z^{es}$$

and hence the normal component E_z^{em} of the electromagnetic solution is generally discontinuous in agreement with the cold-plasma case.

From the WKB solutions (2.52), (2.53) and the boundary conditions (2.87b), (3.20) one readily finds the matching relations

$$\sqrt{\varepsilon_1}\,\eta_0 = \sum_{j=1}^{2} \left\{ (-1)^j \sqrt{\varepsilon_j}\,\eta_j \right\} \qquad (3.21a)$$

$$\frac{\sqrt{\varepsilon_1 - \alpha_0^2}}{\sqrt{\varepsilon_1}}\,\eta_0 = \sum_{j=1}^{2} \left\{ \frac{\sqrt{\varepsilon_j - \alpha_0^2}}{\sqrt{\varepsilon_j}}\,\eta_j - (-1)^j \frac{\alpha_0}{\sqrt{\varepsilon_j/\beta^2}}\,\gamma_j \right\} \qquad (3.21b)$$

$$\frac{\alpha_0}{\sqrt{\varepsilon_1}}\,\eta_0 = \sum_{j=1}^{2} \left\{ (-1)^j \frac{\alpha_0}{\sqrt{\varepsilon_j}}\,\eta_j + \frac{\sqrt{\varepsilon_j/\beta^2 - \alpha_0^2}}{\sqrt{\varepsilon_j/\beta^2}}\,\gamma_j \right\} \qquad (3.21c)$$

$$\frac{\alpha_0\sqrt{\varepsilon_1 - \alpha_0^2}}{\sqrt{\varepsilon_1}}\,\eta_0 = \sum_{j=1}^{2} \left\{ \frac{\alpha_0\sqrt{\varepsilon_j - \alpha_0^2}}{\sqrt{\varepsilon_j}}\,\eta_j + (-1)^j \frac{\varepsilon_j/\beta^2 - \alpha_0^2}{\sqrt{\varepsilon_j/\beta^2}}\,\gamma_j \right\} \qquad (3.21d)$$

where η refers to an electromagnetic, γ to an electrostatic solution, and the index 0 to the incident primary wave. Solving (3.21) for η_1 yields the reflection coefficient

$$R = \left|\frac{\eta_1}{\eta_0}\right|^2 = \left|\frac{A^-}{A^+}\right|^2 \qquad (3.22)$$

with

$$A^\pm = \left\{ \left(\varepsilon_1/\beta^2 - \alpha_0^2\right)^{1/2} \left(\varepsilon_2/\beta^2 - \alpha_0^2\right)^{1/2} + \alpha_0^2 \right\} B^\pm$$

$$\pm \alpha_0^2 \left(\varepsilon_1 - \varepsilon_2\right) \left\{ \left(\frac{\varepsilon_1}{\beta^2} - \alpha_0^2\right)^{1/2} - \left(\frac{\varepsilon_2}{\beta^2} - \alpha_0^2\right)^{1/2} \right\}$$

$$B^\pm = \varepsilon_2 \left(\varepsilon_1 - \alpha_0^2\right)^{1/2} \pm \varepsilon_1 \left(\varepsilon_2 - \alpha_0^2\right)^{1/2}$$

At normal incidence ($\alpha_0^2 = 0$) as well as for cold plasmas ($\beta^2 = 0$) (3.22) simplifies to the well-known Fresnel formula[2]:

$$R = |B^- / B^+|^2$$

At grazing incidence ($\alpha_o^2 = \varepsilon_1$) we have

$$B^- = - B^+$$

and therefore always total reflection R = 1.

Let us now consider a transition to an overdense plasma region

$$\varepsilon_1 > \alpha_o^2 \quad ; \quad \varepsilon_2 < 0$$

In this case (3.22) can be rewritten as

$$R = \frac{(a-b)^2 + (c-d)^2}{(a+b)^2 + (c+d)^2} \quad ; \quad A = 1 - R \tag{3.23}$$

with

$$a = \sqrt{\tau}\ \{ u \hat{u} \hat{w} - (1+\tau) \beta \alpha_1^2 \hat{w} \}$$

$$b = \sqrt{\tau}\ \beta^2 \alpha_1^2 w$$

$$c = (1+\tau) \beta \alpha_1^2 \hat{u} - \tau \beta^2 \alpha_1^2 u$$

$$d = w \hat{w} \hat{u} \quad ; \quad \alpha_1^2 = \alpha_o^2 / \varepsilon_1$$

$$w = \sqrt{1 - \alpha_1^2} \quad ; \quad \hat{w} = \sqrt{1 - \beta^2 \alpha_1^2}$$

$$u = \sqrt{1 + \tau \alpha_1^2} \quad ; \quad \hat{u} = \sqrt{1 + \tau \beta^2 \alpha_1^2}$$

For the general behaviour of this solution we refer to Fig. 9, where the absorption coefficient is evaluated for various parameter values. We indicate the excellent agreement with the thin-layer asymptote described in Fig. 8.

We now restrict our attention to the limits $\beta^2 \to 1$ and $\tau \to 0$ because of the high absorption values observed here. For these cases (3.23) reduces to

$$R = \begin{cases} \left(\dfrac{w \hat{w} - \beta \alpha_1^2}{w \hat{w} + \beta \alpha_1^2} \right)^2 & \tau = 0 \\ (1 - 2 \alpha_1^2)^2 & \beta^2 = 1 \end{cases} \tag{3.24}$$

where (3.24) has a zero at the angle

$$\alpha_1^2 = 1 / (1 + \beta^2) \tag{3.25}$$

This result is conveniently interpreted by the model of an infinitely high potential barrier[49]. In the limit $\tau \to 0$ we find from (3.21a), (3.21d)

$$\eta_2 = 0 \quad , \quad \gamma_2 = 0$$

while (3.21b), (3.21c) then reduce to the boundary condition

$$\underline{E} = 0 \tag{3.26}$$

corresponding to a node in the wave field. Let us now assume an electro-

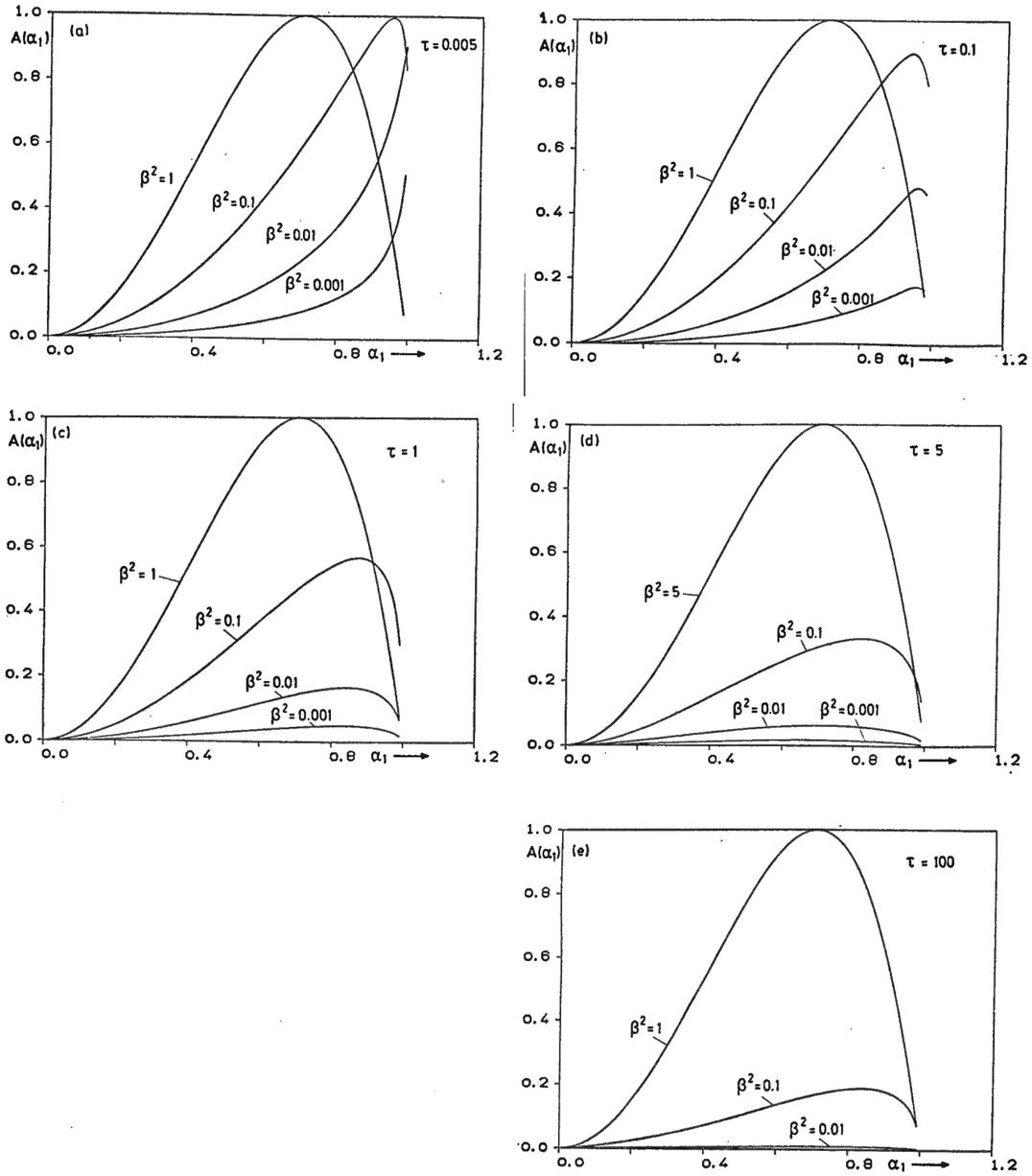


Fig. 9: Absorption from thin layers between undercritical and overcritical plasma regions. The absorption coefficient depends on the layer symmetry $\tau = \varepsilon_1 / (-\varepsilon_2)$ and the temperature β^2. In the limit $\tau \ll 1$ absorption is considerable even at low temperatures (a).

magnetic and an electrostatic wave in medium 1. Condition (3.26) can then only be satisfied if they propagate in directions which are perpendicular to each other (Fig. 10):

$$\Theta_1^{es} = \Theta_1^{em} - \frac{\pi}{2} \tag{3.27}$$

From the refraction laws (2.52a), (2.53a), one has

$$\sin^2 \Theta_1^{em} = \alpha_1^2 \quad ; \quad \sin^2 \Theta_1^{es} = \beta^2 \alpha_1^2 \tag{3.28}$$

Inserting (3.27) into (3.28) again leads to the angle (3.25)

$$\alpha_1^2 (1+\beta^2) = \sin^2 \Theta_1^{em} + \sin^2 \left(\Theta_1^{em} - \frac{\pi}{2} \right) = 1$$

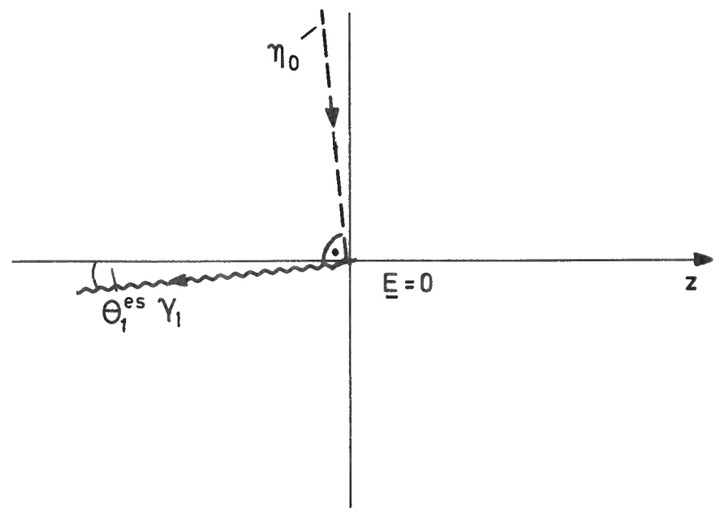


Fig. 10: Geometry for complete mode conversion at thin layers with $\tau \ll 1$. The rays of the incident electromagnetic and the emitted electrostatic waves are perpendicular to each other, and their electric fields add to zero at the interface

4.1 Nonlinear solution ($\beta^2 = 0$)

The nonlinear theory of resonance absorption is commonly based on the capacitor model[26,34,36,37]. In this model the indefinite growth of the resonant oscillations leads to wave-breaking if a critical amplitude is exceeded for which neighbouring fluid elements penetrate. We now derive this model from basic equations and then determine the wave-breaking condition for steady-state solutions.

We consider the model of cold electrons and fixed ions of density n_i governed by the equations

$$(\partial_t + \underline{v}\cdot\nabla + \nu)\,\underline{v} = \frac{q}{m}\left(\underline{E} + \frac{1}{c}\,\underline{v}\times\underline{B}\right) \tag{4.1}$$

$$\nabla\times\underline{B} = \frac{1}{c}\,\partial_t\,\underline{E} + \frac{4\pi q}{c}\,n\,\underline{v} \tag{4.2}$$

$$\nabla\times\underline{E} = -\frac{1}{c}\,\partial_t\,\underline{B} \tag{4.3}$$

$$\nabla\cdot\underline{E} = 4\pi q\,(n - n_i) \tag{4.4}$$

For convenience we eliminate $\underline{E}$ and n by using (4.1) and (4.4) respectively and introduce dimensionless variables by

$$\tilde{t} = \omega t \;;\quad \tilde{\underline{x}} = k_0\,\underline{x} \;;\quad \tilde{\underline{v}} = v_{os}^{-1}\,\underline{v} \;;\quad \tilde{\nabla} = k_0^{-1}\,\nabla \;;$$

$$\tilde{\underline{B}} = \{\alpha_0\,|B_x(0)|\}^{-1}\,\underline{B} \;;\quad \tilde{\omega}_p = \omega^{-1}\,\omega_p \;;\quad \tilde{\nu} = \omega^{-1}\,\nu \tag{4.5}$$

where

$$\omega_p^2 = \frac{4\pi q^2 n_0}{m} \quad ; \quad v_{os} = \nu^{-1}\,\frac{q\,\alpha_0\,|B_x(0)|}{m}$$

Then $\tilde{\underline{v}}$ and $\tilde{\underline{B}}$ are determined by the system

$$\underline{\underline{L}}\cdot\left(\frac{D}{Dt} + \tilde{\nu}\right)\tilde{\underline{v}} + \tilde{\omega}_p^2\;\tilde{\underline{v}} =$$

$$= \tilde{\nu}\left\{\tilde{\nabla}\times\tilde{\underline{B}} + p\,\underline{\underline{L}}\cdot\tilde{\underline{v}}\times\tilde{\underline{B}}\right\} \tag{4.6}$$

$$\nabla\times\left(\frac{D}{Dt} + \tilde{\nu}\right)\tilde{\underline{v}} = -\,\tilde{\nu}\left\{\partial_t\,\tilde{\underline{B}} - p\,\tilde{\nabla}\times(\tilde{\underline{v}}\times\tilde{\underline{B}})\right\} \tag{4.7}$$

where

$$\underline{\underline{L}} = \frac{D}{Dt}\,\underline{\underline{I}} + p\,(\underline{v}\otimes\tilde{\nabla} - \underline{v}\cdot\tilde{\nabla}\,\underline{\underline{I}})$$

$$\frac{D}{Dt} = \partial_{\tilde{t}} + p\,\tilde{\underline{v}}\cdot\tilde{\nabla} \quad ; \quad p = v_{os}/c$$

We now restrict our attention to the ansatz

$$\tilde{\underline{B}} = (B_x, 0, 0) \quad ; \quad \tilde{\underline{v}} = (0, v_y, v_z) \tag{4.8}$$

and assume the scaling laws (2.70)

$$v_z = O(1) \quad ; \quad v_y = O(\tilde{\nu}\log\tilde{\nu}) \quad ; \quad B_x = O(1)$$

$$\partial_{\tilde{z}}\,v_z = O(\tilde{\nu}^{-1}) \;;\quad \partial_{\tilde{z}}\,v_y = O(1) \quad ; \quad \partial_{\tilde{z}}\,B_x = 0 \tag{4.9}$$

with

$$\partial_{\tilde{x}} = 0 \quad ; \quad \partial_{\tilde{y}} = O(1) \quad ; \quad \partial_{\tilde{z}} = O(\tilde{\nu}^{-1})$$

for all other derivatives. With (4.8), (4.9) the individual terms of the z component of (4.6) are easily estimated:

$$\frac{D^2}{Dt^2} V_z = \frac{d^2}{dt^2} V_z + O(pv_y) + O(p^2 v_y \partial_{\tilde{z}} v_z)$$

$$\left\{ p(\tilde{\underline{v}} \otimes \tilde{\nabla} - \tilde{\nabla}\cdot\tilde{\underline{v}}\, \underline{\underline{I}})\cdot \frac{D}{Dt}\tilde{\underline{v}} \right\}_z = O(pv_y) + O(p^2 v_y \partial_{\tilde{z}} v_z)$$

$$\left\{ \tilde{\nu}\, \underline{\underline{L}}\cdot\tilde{\underline{v}} \right\}_z = \tilde{\nu}\frac{d}{dt} v_z + O(\tilde{\nu} p v_y) + O(\tilde{\nu} v_y) \qquad (4.10)$$

$$\left\{ p \frac{D}{Dt}\tilde{\underline{v}} \times \tilde{\underline{B}} \right\}_z = -p^2 v_z \partial_{\tilde{z}}(v_y B_x) + O(pv_y)$$

$$\left\{ p^2(\tilde{\underline{v}} \otimes \tilde{\nabla} - \tilde{\underline{v}}\cdot\tilde{\nabla}\,\underline{\underline{I}})\cdot\tilde{\underline{v}}\times\tilde{\underline{B}} \right\}_z = p^2 v_z \partial_{\tilde{y}}(v_z B_x) + O(p^2 v_y^2)$$

where

$$\frac{d}{dt} = \partial_{\tilde{t}} + p v_z \partial_{\tilde{z}}$$

Assuming further

$$p = O(\tilde{\nu}) \qquad (4.11)$$

which will be verified by the result, and keeping terms consistent up to the order $O(\tilde{\nu})$, one gets from (4.6), (4.10), (4.11) in the limit $\tilde{\nu} \longrightarrow 0$:

$$\frac{d^2}{dt^2} V_z + \tilde{\nu}\frac{d}{dt} V_z + \tilde{\omega}_p^2 V_z = -\tilde{\nu}\,\partial_{\tilde{y}} B_x \qquad (4.12)$$

Introducing Lagrangian coordinates z_0, τ:

$$\tilde{z} = z_0 + p\,\delta(z_0,\tau) \quad ; \quad \tilde{t} = \tau \qquad (4.13)$$

$$\delta(z_0,\tau) = \int_0^\tau d\tau'\, V(z_0,\tau')$$

$$V(z_0,\tau) = v_z(\tilde{z}[z_0,\tau]\,,\,\tilde{t}[\tau])$$

and integrating with respect to τ (4.12) can equivalently be written as

$$\partial_\tau^2 \delta + \tilde{\nu}\,\partial_\tau \delta + F(\delta) = \tilde{\nu}\, D(\tau,\delta) \qquad (4.14)$$

with

$$F(\delta) = \frac{1}{p}\int_{z_0}^{z_0+p\delta} dz\; \tilde{\omega}_p^2(z) \quad ; \quad D(\tau,\delta) = -\int_0^\tau d\tau'\, \partial_{\tilde{y}} B_x$$

Equation (4.14) represents the equation of motion for an electron with initial position z_0 under the influence of an electrostatic restoring force $F(\delta)$ and an electromagnetic driver $D(\tau,\delta)$. Since the electron displacement (4.11) has been assumed to be small, these forces can be expanded about the equilibrium position z_0, yielding

$$F(\delta) = \omega_p^2(z_0)\,\delta + \tfrac{1}{2}\omega_p^{2\prime}(z_0)\, p\,\delta^2 + O(\tilde{\nu}^2)$$

$$D(\tau,\delta) = e^{i\{\psi + \alpha_0 \tilde{y}\}}\left(e^{-i\tau} - 1\right) + O(\tilde{\nu}^2) \qquad (4.15)$$

with

$$B_x = \frac{1}{\alpha_0}\, e^{i\{\gamma + \alpha_0 \tilde{y} - \tau\}} \quad ; \quad \partial_{\tilde{z}} B_x = 0$$

The anharmonic term here is of the same order as the damping term and the driver term, but for simplicity we shall limit the following discussion to the model of a pure harmonic force. Then looking for the time asymptotic solution ($\tau \to \infty$), one gets from (4.14), (4.15)

$$\delta(z_0,\tau) = \frac{\tilde{\nu}\, e^{i\varphi(z_0)}}{\{(1-\tilde{\omega}_p^2)^2 + \tilde{\nu}^2\}^{1/2}}\, D(\tau,0)$$

$$\operatorname{tg}\varphi(z_0) = \frac{-\tilde{\nu}}{1-\tilde{\omega}_p^2}$$

Assuming, furthermore, a linear profile

$$1-\tilde{\omega}_p^2 = -z_0/\tilde{L} \quad ; \quad \tilde{L} = k_0 L$$

yields the real displacement

$$\delta(\xi_0,\tau) = \frac{1}{\sqrt{1+\xi_0^2}}\cos(\gamma + \alpha_0\tilde{y} + \varphi(\xi_0) - \tau) \qquad (4.16)$$

$$\varphi(\xi_0) = \begin{cases} \operatorname{arctg}\,\xi_0^{-1} & ; \quad \xi_0 \geq 0 \\ \operatorname{arctg}\,\xi_0^{-1} + \pi & ; \quad \xi_0 \leq 0 \end{cases}$$

$$\xi_0 = z_0/(\tilde{L}\tilde{\nu})$$

We now discuss the wave-breaking condition as given by the steady-state

solution (4.16). In order that the coordinate transformation (4.13) have a unique inversion, it is necessary that

$$\frac{\partial\xi}{\partial\xi_0} > 0 \quad ; \quad \xi = \xi_0 + \frac{p}{\tilde{L}\tilde{\nu}}\,\delta \qquad (4.17)$$

Differentiating (4.16), we get

$$\frac{\partial\delta}{\partial\xi_0} = (1+\xi_0)^{-3/2}\{\sin(\gamma+\varphi-\tau) - \xi_0\cos(\gamma+\varphi-\tau)\}$$

$$\geqslant \frac{-1}{1+\xi_0^2} \geqslant -1 \qquad (4.18)$$

where the equality sign is valid once in each oscillation period at $\xi_0 = 0$. From (4.17), (4.18) one readily finds

$$p < \tilde{L}\tilde{\nu} \qquad (4.19)$$

In this form condition (4.19) just requires that the maximum electron displacement

$$p = k_0\left(\frac{v_{os}}{\omega}\right)$$

remain smaller than the resonance width

$$\tilde{L}\tilde{\nu} = (k_0 L)\left(\frac{\nu}{\omega}\right)$$

It is easily rewritten in the more usual form

$$v_{os} < \sqrt{v_0\,\omega\,L} \quad ; \quad v_0 = \frac{q\alpha_0 |B_x(0)|}{m\omega} \qquad (4.20)$$

Comparing (4.20) with the wave-breaking condition derived in previous work[26,50] for non-stationary solutions

$$v_{os} < \sqrt{2 v_0\,\omega\,L}$$

we find the wave-breaking amplitude here to be a factor of $\sqrt{2}$ lower.

We finally present the nonlinear solution to the resonance field

$$E_z = \frac{\alpha_0 |B_x(0)|}{\tilde{\nu}} E(\xi,\tau) \; ; \quad E(\xi,\tau) = -\delta(\xi,\tau)$$

in Fig. 11. For small amplitudes the linear solution is seen (a), while at large amplitudes steepening of the wave leads to wave-breaking (b), (c).

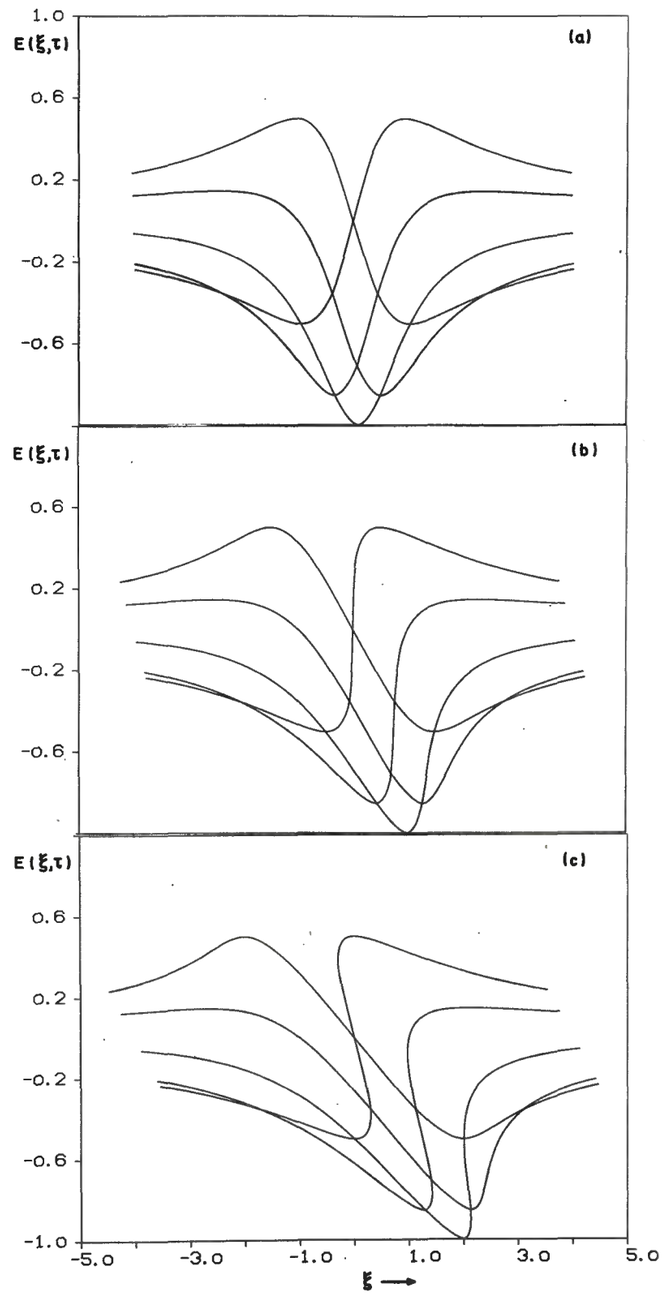


Fig. 11: Spatial and temporal behaviour of the nonlinear resonance field. In each picture the time interval between successive curves is $(\pi/4)\omega^{-1}$. The nonlinear distortion of the field leads to wave breaking if the maximum electron displacement p becomes larger than the resonance width $\tilde{\nu}\tilde{L}$.
a) $p/(\tilde{\nu}\tilde{L}) = 0.1$, b) $p/(\tilde{\nu}\tilde{L}) = 1$, c) $p/(\tilde{\nu}\tilde{L}) = 2$.

4.2 Magnetized plasmas ($\beta^2 = 0$)

In magnetized plasmas wave propagation is complicated by the appearance of new modes and resonances[3]. The principal features, such as circularly polarized waves and hybrid waves, are now discussed in terms of the cold-plasma model. We shall establish a condition for magnetic field effects to be small.

Let us derive the wave equation for cold, magnetized plasmas. Using (2.4), (2.5) and the equation of motion

$$(-i\omega + \nu)\, \underline{v}_1 = \frac{q}{m}\left(\underline{E}_1 + \frac{1}{c}\, \underline{v}_1 \times \underline{B}_0\right)$$

we find

$$i\, \nabla \times \underline{B}_1 = k_0 \left\{ \gamma_1\, \underline{\underline{I}} + \gamma_2\, \underline{\underline{K}} \right\} \cdot \underline{E}_1 \qquad (4.21a)$$

$$\nabla \times \underline{E}_1 = i k_0\, \underline{B}_1 \qquad (4.21b)$$

with

$$\gamma_1 = 1 - \frac{\tau\, \omega_p^2/\omega^2}{1 - (\tau\, \omega_c/\omega)^2}$$

$$\gamma_2 = \frac{\frac{\omega_c}{\omega}\left(\tau\, \frac{\omega_p}{\omega}\right)^2}{1 - (\tau \omega_c/\omega)^2}$$

$$\underline{\underline{K}} \cdot \underline{E}_1 = i\, \hat{\underline{B}} \times \underline{E}_1 + \tau\, \frac{\omega_c}{\omega}\, (\underline{E}_1 \cdot \hat{\underline{B}})\, \hat{\underline{B}}$$

$$\omega_c = \frac{q B_0}{mc} \quad ; \quad \tau = \frac{1}{1 + i\frac{\nu}{\omega}} \quad ; \quad \underline{B}_0 = B_0\, \hat{\underline{B}} \quad ; \quad \hat{\underline{B}}^2 = 1$$

The general problem (4.21) considerably simplifies if the wave propagation is chosen parallel or perpendicular to the magnetic field direction. The first case can only be realized for normal incidence. Requiring therefore

$$\partial_x = 0 \quad ; \quad \partial_y = 0 \quad ; \quad \hat{\underline{B}} = \underline{e}_z$$

the system (4.21) reduces to

$$E_x'' + \gamma_1 E_x - i\gamma_2 E_y = 0 \qquad (4.21a)$$

$$E_y'' + \gamma_1 E_y + i\gamma_2 E_x = 0 \qquad (4.21b)$$

$$(\gamma_1 + \tau\, \frac{\omega_c}{\omega}\, \gamma_2)\, E_z = 0 \qquad (4.21c)$$

Here it is seen that the longitudinal and transverse wave components are decoupled. From (4.21c) we find the dispersion relation of the longitudinal mode

$$\gamma_1 + \tau\, \frac{\omega_c}{\omega}\, \gamma_2 = 1 - \tau\, \frac{\omega_p^2}{\omega^2} = \varepsilon = 0$$

which is not changed by the presence of the magnetic field. Introducing now circularly polarized waves by

$$E_{R/L} = E_x \pm i E_y$$

(4.21a), (4.21b) yield readily

$$E''_{R/L} + (\gamma_1 \mp \gamma_2) E_{R/L} = 0 \qquad (4.22a)$$

$$\gamma_1 \mp \gamma_2 = 1 - \frac{\tau \omega_p^2}{\omega(\omega \mp \tau \omega_c)} \qquad (4.22b)$$

In the collisionless limit the dielectric constant (4.22b) describes a resonance at the electron cyclotron frequency ω_c and zeros at

$$\omega_{1/2} = \frac{\pm \omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}}{2} \qquad (4.23)$$

In weakly magnetized plasmas with low collision frequency one has

$$\gamma_1 \mp \gamma_2 \longrightarrow 1 - \frac{\omega_p^2}{\omega^2}\left(1 - i\frac{\nu}{\omega} \pm \frac{\omega_c}{\omega}\right)$$

$$\nu/\omega << 1$$

$$\omega_c/\omega << 1$$

Hence we neglect the magnetic field terms by the condition

$$\frac{\nu}{\omega} >> \frac{\omega_c}{\omega} \qquad (4.24)$$

Let us now consider wave propagation perpendicular to the magnetic field. Here the only possible configuration is

$$\partial_x = 0 \; ; \quad \hat{\underline{B}} = \underline{e}_x \qquad (4.25)$$

From (4.21), (4.25) one gets the wave equation

$$E''_x + k_0^2 (\varepsilon - \alpha_0^2) E_x = 0 \qquad (4.26a)$$

for s-polarization and

$$B''_x - \frac{\gamma'}{\gamma} B'_x + \left\{ k_0^2(\gamma - \alpha_0^2) + k_0 \alpha_0 \left(\gamma_0' - \gamma' \frac{\gamma_0}{\gamma}\right) \right\} B_x = 0$$

$$\gamma_0 = \gamma_2 / \gamma_1 \; ; \quad \gamma = \gamma_1 - \gamma_0 \gamma_2 \qquad (4.26b)$$

for p-polarization. Magnetic field corrections are only present in (4.26b). For normal incidence ($\alpha_0 = 0$) they can be expressed by an effective dielectric constant

$$\gamma = \frac{(\gamma_1 + \gamma_2)(\gamma_1 - \gamma_2)}{\gamma_1} = \qquad (4.27)$$

$$= \frac{\left(1 - \frac{\tau \omega_p^2}{\omega(\omega - \tau\omega_c)}\right)\left(1 - \frac{\tau \omega_p^2}{\omega(\omega + \tau\omega_c)}\right)}{1 - \frac{\tau \omega_p^2/\omega^2}{1 - (\tau\omega_c/\omega)^2}}$$

In the collisionless limit (4.27) describes a resonance at the upper hybrid frequency

$$\omega_H = (\omega_p^2 + \omega_c^2)^{1/2}$$

and zeros $\omega_{1/2}$ as given by (4.23). For weakly magnetized plasmas with a small collision frequency one obtains

$$\gamma \xrightarrow[\substack{\frac{\omega_c}{\omega} << 1}]{\frac{\nu}{\omega} << 1} \frac{\left\{1 - \frac{\omega_p^2}{\omega^2}\left(1 - i\frac{\nu}{\omega} + \frac{\omega_c}{\omega}\right)\right\}\left\{1 - \frac{\omega_p^2}{\omega^2}\left(1 - i\frac{\nu}{\omega} - \frac{\omega_c}{\omega}\right)\right\}}{1 - \frac{\omega_p^2}{\omega^2}\left(1 - i\frac{\nu}{\omega} + \left(\frac{\omega_c}{\omega}\right)^2\right)}$$

Hence magnetic field terms can again be neglected by (4.24):

$$\gamma \longrightarrow \mathcal{E} \qquad (\omega_c << \nu)$$

Finally, we consider (4.26b) for oblique incidence.
Introducing

$$y = Bx / \sqrt{\mathcal{E}}$$

and assuming

$$\omega_c << \nu << \omega$$

we find

$$y'' + \left\{ k_0^2 (\mathcal{E} - \alpha_0^2) - \frac{3}{4}\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)^2 + \frac{1}{2}\frac{\mathcal{E}''}{\mathcal{E}} - k_0 \alpha_0 \frac{\omega_c}{\omega}\sqrt{\mathcal{E}}\left(\frac{\mathcal{E}'}{\mathcal{E}} - 2\frac{\mathcal{E}'}{\mathcal{E}^2}\right)\right\} y = 0$$

The remaining magnetic field term is seen to be small if

$$\frac{\omega_c}{\omega} << \frac{1}{k_0 \sqrt{\mathcal{E}}\, \alpha_0 L} \qquad (4.28)$$

This condition is usually much weaker than (4.24).

4.3 Linear Landau damping

To estimate the distance over which the electrostatic wave can freely propagate, we evaluate the well-known dispersion relation of the linear Vlasov theory[3]:

$$D(\omega, k) = 1 - \frac{\omega_p^2}{k^2} \int dv_3 \, \frac{F'(v_3)}{v_3 - \omega/k} = 0 \qquad (4.29)$$

$$F(v_3) = \int dv_1 \int dv_2 \, f_0 \quad ; \quad \underline{k} = (0, 0, k)$$

Generally, (4.29) relates a complex wave number to a complex frequency:

$$k = k_r + i k_i$$

$$\omega = \omega_r + i \omega_i$$

If we assume small imaginary parts

$$\left|\frac{\omega_i}{\omega_r}\right| << 1 \quad ; \quad \left|\frac{k_i}{k_r}\right| << 1$$

and expand (4.29) about ω_r and k_r, we get

$$D_r = 0 \qquad (4.30)$$

$$D_i + \omega_i \partial_{\omega_r} D_r + k_i \partial_{k_r} D_r = 0$$

where

$$D_r = 1 - \frac{\omega_p^2}{k^2} P \int dv_3 \, \frac{F'(v_3)}{v_3 - \omega_r / k_r}$$

$$D_i = - \frac{\omega_p^2}{k_r^2} \pi F'(\omega_r/k_r)$$

and $P\int dv_3$ denotes the Cauchy principal part. To apply (4.30) to linear mode conversion, we take the frequency as given

$$\omega_r = \omega \quad ; \quad \omega_i = 0$$

assume a Maxwellian distribution

$$F(v_3) = \left(\frac{m}{2\pi T}\right)^{1/2} exp\left\{- \frac{m v_3^2}{2T}\right\}$$

and use the fluid approximation

$$\left(\frac{s k_r}{\omega_r}\right) << 1$$

One then readily obtains

$$D_r = \varepsilon - \frac{s^2 k_r^2}{\omega^2}$$

$$D_i = 3^{3/2} \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{1}{\varepsilon}\right)^{3/2} exp\left\{- \frac{3}{2} \frac{1}{\varepsilon}\right\}$$

and

$$k_r = - \frac{k_0 \sqrt{\varepsilon}}{\beta}$$

$$k_i = \frac{-1}{s\sqrt{\varepsilon}} \omega_p \sqrt{\frac{\pi}{8}} \left\{\frac{3}{\varepsilon} exp\left(- \frac{1}{\varepsilon}\right)\right\}^{3/2} \qquad (4.31)$$

We now consider wave propagation from a point z_1 to a point $z < z_1$. In WKB approximation the wave amplitude then decreases by the factor

$$exp\{-\Psi\} \; ; \quad \Psi = \int_{z_1}^{z} dz \; k_i = - \int_{z}^{z_1} dz \; k_i \qquad (4.32)$$

Assuming a linear profile

$$\varepsilon = - z/L \; ; \qquad \xi = \left(\frac{k_0 L}{\beta}\right)^{2/3} (-\varepsilon)$$

we easily find from (4.31), (4.32)

$$\Psi = \sqrt{\frac{3}{2}\pi} \frac{k_0 L}{\beta} \left\{ f(\xi) - f(\xi_1) \right\} \qquad (4.33)$$

$$f(\xi) = exp\left\{ - \frac{3}{2} \left(\frac{k_0 L}{\beta}\right)^{2/3} \frac{1}{(-\xi)} \right\}$$

For numerical evaluation of (4.33) we refer to Table 1. We note that the contribution from the lower integration boundary, because of assumption (2.33b), is always small if we choose

$$\xi_1 \approx - 1$$

Hence if it is neglected, (4.33) simplifies to

$$\Psi = \sqrt{\frac{3}{2}\pi} \left(\frac{k_0 L}{\beta}\right) exp\left\{ - \frac{3}{2} \frac{1}{\varepsilon} \right\} \qquad (4.34)$$

At a fixed density the damping argument Ψ is seen to be a linear function of $(k_0 L/\beta)$ only.

$(\beta/k_0 L)^{2/3}$	$(-\xi)$	ε	$f(\xi)$	ψ
0.01	1	0.01	7×10^{-66}	0
	5	0.05	9×10^{-14}	$1{,}9 \times 10^{-10}$
	10	0.1	3×10^{-7}	6×10^{-4}
	15	0.15	4×10^{-5}	9×10^{-2}
	20	0.2	$5{,}5 \times 10^{-4}$	1,2
	25	0.25	$2{,}5 \times 10^{-3}$	5,4
	30	0.3	7×10^{-3}	14,6
0,03	1	0.03	$1{,}9 \times 10^{-22}$	0
	5	0.15	$4{,}5 \times 10^{-5}$	$1{,}8 \times 10^{-2}$
	10	0.3	7×10^{-3}	2,9
0.06	1	0.06	$1{,}4 \times 10^{-11}$	0
	2	0.12	$3{,}7 \times 10^{-6}$	$5{,}46 \times 10^{-4}$
	3	0.18	$2{,}4 \times 10^{-4}$	$3{,}5 \times 10^{-2}$
	4	0.24	$1{,}9 \times 10^{-3}$	$2{,}8 \times 10^{-1}$
	5	0.3	$6{,}7 \times 10^{-3}$	1,0
0.09	1	0.09	$5{,}8 \times 10^{-8}$	0
	1,5	0.13	$1{,}5 \times 10^{-5}$	$1{,}2 \times 10^{-3}$
	2	0.18	$2{,}4 \times 10^{-4}$	$1{,}9 \times 10^{-2}$
	2,5	0.22	$1{,}27 \times 10^{-3}$	0,1
	3	0.27	$3{,}87 \times 10^{-3}$	0,31
	3,5	0.31	$8{,}55 \times 10^{-3}$	0,69

Table 1: Values of ψ

4.4 Relativistic temperatures

We are now concerned with the relativistic form of the fluid model established in Sec. 2.1. We consider a relativistic ideal Boltzmann gas with three degrees of freedom subject to the conservation laws of an adiabatic and formally viscous fluid. From this model we derive the wave equations and then discuss the thermodynamic properties of the ideal gas. According to (2.33) one can expect the relativistic fluid model to be valid if

$$\varepsilon << 1 \quad ; \quad (k_0 L)^{2/3} >> 1$$

In the following all velocities will be expressed in units of the speed of light.

Let us consider the relativistic conservation laws of particle number and 4-momentum[50]

$$(n u^\mu)_{,\mu} = 0 \tag{4.35}$$

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{4.36}$$

where

$$u^\mu = \gamma (1, \vec{v}) \quad ; \quad \gamma = (1 - v^2)^{-1/2} \tag{4.37}$$

denotes the 4-velocity,

$$q F^\mu = -q u_\nu \{ A^{\mu,\nu} - A^{\nu,\mu} \} \tag{4.38}$$

$$= q \gamma (\vec{v} \cdot \vec{E} , \vec{E} + \vec{v} \times \vec{B})$$

with

$$A^\mu = (A^\circ, \vec{A}) \quad ; \quad \phi_{,\mu} = (\partial_t \phi, \vec{\nabla}\phi)$$

$$\vec{E} = -\partial_t \vec{A} - \vec{\nabla} A_\circ \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

the electromagnetic 4-force,

$$T^{\mu\nu} = (\varrho + p) u^\mu u^\nu + p\, g^{\mu\nu} + \tau^{\mu\nu}$$

$$\tau^{\mu\nu} = -\eta \left\{ u^{\mu,\nu} + u^{\nu,\mu} + u^\alpha (u^\mu u^\nu)_{,\alpha} - \right. \tag{4.39}$$

$$\left. - \tfrac{2}{3} (u^\alpha{}_{,\alpha}) (g^{\mu\nu} + u^\mu u^\nu) \right\}$$

$$\eta = p/(-i\omega)$$

the energy-momentum tensor of the adiabatic viscous fluid with a formal complex collision frequency and

$$g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{4.40}$$

the Lorentz metric. The particle density $n(x^\mu)$, pressure $p(x^\mu)$ and total energy dencity $\varrho(x^\mu)$ are defined as scalar quantities, assuming the values as measured in the rest frame of the fluid element at the event x^μ.

Equation (4.36) is conveniently separated into its components parallel and

orthogonal to the 4-velocity:

$$u^\mu \left\{ \varrho_{,\mu} - \frac{\varrho + p}{n} n_{,\mu} - \tau_{\mu\nu}{}^{,\nu} \right\} = 0 \tag{4.41}$$

$$(\varrho + p) u^\nu u^\mu{}_{,\nu} = q n F^\mu - \tag{4.42}$$

$$- (g^{\mu\nu} + u^\mu u^\nu)(p g_\nu{}^\alpha + \tau_\nu{}^\alpha)_{,\alpha}$$

The fluid model (4.35), (4.41), 4.42) has to be supplemented by an equation of state

$$p = p(\varrho, n) \tag{4.43}$$

which is discussed below.

We now derive the wave equations, assuming the perturbation ansatz

$$a^\mu = \bar{a}^\mu + \hat{a}^\mu \tag{4.44}$$

with

$$\bar{a}^\mu{}_{,0} = 0 \qquad \hat{a}^\mu \propto e^{-i\omega t}$$

$$\bar{u}^\mu = (1, 0) \qquad \bar{A}^\mu = (\bar{A}^\circ, 0)$$

and neglecting gradient terms as discussed in Sec. 2.1. Then (4.41) reduces to the adiabatic law:

$$\hat{\varrho} = \frac{\bar{\varrho} + \bar{p}}{\bar{n}} \hat{n} \tag{4.45}$$

and the time component of the 4-velocity vanishes:

$$u^\mu \, (u^\nu u_\nu)_{,\mu} = 0 \qquad (4.46)$$

$$\bar{u}^\nu \, \bar{u}^\mu \, \hat{u}_{\nu,\mu} = 0 \quad ; \quad \hat{u}^0 = 0$$

From (4.37), (4.46) it is seen that all relativistic velocity terms are neglected by linearization. The remaining components of the 4-velocity then follow from (4.42), (4.43), (4.45), 4.46):

$$(-i\omega) \, \frac{\bar{\varepsilon} + \bar{p}}{\bar{n}} \, \hat{\vec{u}} = q \hat{\vec{E}} -$$

$$- \frac{1}{i\omega} \left\{ \left(\left.\frac{\partial p}{\partial n}\right|_\varepsilon + \frac{\bar{\varepsilon}+\bar{p}}{\bar{n}} \left.\frac{\partial p}{\partial \varepsilon}\right|_n + \frac{4}{3} \frac{\bar{p}}{\bar{n}} \right) \vec{\nabla} (\vec{\nabla} \cdot \hat{\vec{u}}) - \right.$$

$$\left. - \frac{\bar{p}}{\bar{n}} \, \vec{\nabla} \times (\vec{\nabla} \times \hat{\vec{u}}) \right\} \quad ; \quad u^\mu = (u^0, \vec{u})$$

yielding

$$\frac{4\pi i}{\omega} \hat{\vec{j}} = - \frac{\omega_{pr}^2}{\omega^2} + \frac{1}{\omega^2 m_r} \left\{ \left(\left.\frac{\partial p}{\partial n}\right|_\varepsilon + m_r \left.\frac{\partial p}{\partial \varepsilon}\right|_n + \right.\right.$$

$$\left.\left. + \frac{4}{3} \frac{\bar{p}}{\bar{n}} \right) \vec{\nabla} \vec{\nabla} \cdot \hat{\vec{E}} - \frac{\bar{p}}{\bar{n}} \, \vec{\nabla} \times (\vec{\nabla} \times \hat{\vec{E}}) \right\} \qquad (4.47)$$

with

$$m_r = \frac{\bar{\varepsilon} + \bar{p}}{\bar{n}} \quad ; \quad \omega_{pr}^2 = \frac{4\pi q^2 \bar{n}}{m_r} \quad ; \quad \hat{\vec{j}} = q \bar{n} \hat{\vec{u}}$$

Inserting the current density (4.47) into (2.4), (2.5) one gets

$$\left(1 + \frac{\bar{p}}{m_r \bar{n}}\right) \vec{\nabla} \times \hat{\vec{B}} = - i\omega \left\{ \varepsilon_r \hat{\vec{E}} + \frac{\beta_r^2}{\omega^2} \, \vec{\nabla} (\vec{\nabla} \cdot \hat{\vec{E}}) \right\}$$

$$\vec{\nabla} \times \hat{\vec{E}} = i\omega \hat{\vec{B}} \qquad (4.48)$$

with

$$\varepsilon_r = 1 - \frac{\omega_{pr}^2}{\omega^2} \quad ; \quad \beta_r^2 = \frac{\left.\frac{\partial p}{\partial n}\right|_\varepsilon + m_r \left.\frac{\partial p}{\partial \varepsilon}\right|_n + \frac{4}{3} \frac{\bar{p}}{\bar{n}}}{m_r}$$

Comparing now the relativistic field equations (4.48) with the classical ones (2.20), we find complete formal analogy simply by substituting

$$\varepsilon \longrightarrow \frac{\varepsilon_r}{1 + \bar{p}/m_r \bar{n}}$$

$$\beta^2 \longrightarrow \frac{\beta_r^2}{1 + \bar{p}/m_r \bar{n}} \qquad (4.49)$$

Hence the dispersion relations of electromagnetic and electrostatic waves are obtained as

$$(k^{em})^2 = \omega^2 \frac{\varepsilon_r}{1 + \bar{p}/m_r \bar{n}} \quad ; \quad \omega^2 = \omega_{pr}^2 + \left(1 + \frac{\bar{p}}{m_r \bar{n}}\right) (k^{em})^2$$

(4.50)

$$(k^{es})^2 = \omega^2 \frac{\varepsilon_r}{\beta_r^2} \quad ; \quad \omega^2 = \omega_{pr}^2 + \beta_r^2 \, (k^{es})^2$$

In order to evaluate this result further, one has to specify the equation of state (4.43). We therefore consider an ideal Boltzmann gas in local thermodynamic equilibrium. Each fluid element of volume V is assumed to maintain a mean energy E and a mean particle number N because of its contact with its surroundings. All thermodynamic functions can then be deduced from the grand potential[51]

$$J = -TV\lambda^{-3}\exp\{\mu/T\} \tag{4.51a}$$

$$\lambda^{-3} = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \exp\left(-\frac{\mathcal{E}(p)}{T}\right)$$

$$J = E - TS - \mu N \tag{4.51b}$$

$$dJ = -S\,dT - p\,dV - N\,d\mu \tag{4.51c}$$

where μ, T, $\mathcal{E}(p)$, S denote the chemical potential, the temperature, the single-particle energy and the entropy, respectively. Integrating (4.51a) with

$$\mathcal{E}(p) = (p^2 + m^2)^{1/2}$$

yields in a straightforward manner

$$\lambda^{-3} = \left(\frac{m}{2\pi\hbar}\right)^3 \frac{2\pi^{3/2}}{\Gamma(3/2)}\left(\frac{1}{\alpha^2}K_1(\alpha) - \frac{1}{\alpha}K_1'(\alpha)\right) \tag{4.52}$$

$$\alpha = m/T$$

where $K_1(\alpha)$ denotes the modified Bessel function of the first order with the asymptotic expansion[47,52]

$$K_1(\alpha) \longrightarrow \begin{cases} \alpha^{-1} & ;\ \alpha << 1 \\ \sqrt{\frac{\pi}{2\alpha}}\,e^{-\alpha}\left(1 + \frac{0.375}{\alpha} - \frac{0.117}{\alpha^2} + \cdots\right) & ;\ \alpha >> 1 \end{cases} \tag{4.53}$$

To establish the relation (4.43) between the pressure, energy density and particle density, first we derive from (4.51), 4.52) the expressions

$$TS = -T\left(\frac{\partial J}{\partial T}\right)_{V,\mu} = TN - \mu N + T^2 V e^{\mu/T}\frac{d}{dT}(\lambda^{-3})$$

$$N = -\left(\frac{\partial J}{\partial \mu}\right)_{T,V} = V\lambda^{-3}e^{\mu/T} \tag{4.54}$$

and then, using (4.51b), the energy density

$$\mathcal{E} = n\,e(T) \tag{4.55}$$

$$n = \frac{N}{V}\ ;$$

$$e(T) = \frac{E}{N} = 3T + m\,\frac{\alpha K_1}{K_1 - \alpha K_1'}$$

$$\mathcal{E} \longrightarrow \begin{cases} m + \frac{3}{2}T & ;\ \alpha >> 1 \\ 3T & ;\ \alpha << 1 \end{cases}$$

From the identity

$$\alpha K_\nu' - \nu K_\nu = -\alpha K_{\nu+1}$$

we find (4.55) in the more usual form derived in Ref. 53:

$$e(T) = 3T + m\left(K_1/K_2\right)$$

Finally, from (4.51a), (4.54) and the Duhem-Gibbs relation[51]

$$J = -pV$$

one gets the ideal gas law

$$p = nT \tag{4.56}$$

Then (4.43) follows from (4.55), (4.56) in the implicit form

$$\mathcal{E} = n\,e(p/n) \tag{4.57}$$

We shall now evaluate the thermodynamic functions contained in the field equations (4.48). From

$$\frac{\varepsilon + p}{n} = T + e(T)$$

$$\left.\frac{\partial p}{\partial n}\right|_{\varepsilon} = \frac{d(T/e)}{d(1/e)} = T - \frac{e}{e'}$$

$$\left.\frac{\partial p}{\partial \varepsilon}\right|_{n} = \frac{dT}{de} = \frac{1}{e'}$$

one readily gets the coefficients

$$\varepsilon_r = 1 - \frac{4\pi q^2 \bar{n}}{\omega^2(\bar{T}+e)} \qquad (4.58)$$

$$\longrightarrow \begin{cases} 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{5}{2}\frac{\bar{T}}{m}\right) & ; \quad \frac{T}{m} \ll 1 \\ 1 - \frac{\omega_p^2}{\omega^2}\frac{m}{4\bar{T}} & ; \quad \frac{T}{m} \gg 1 \end{cases}$$

$$\beta_r^2 = \frac{\bar{T}\left(\frac{7}{3} + \frac{1}{e'}\right)}{\bar{T} + e} \qquad (4.59)$$

$$\longrightarrow \begin{cases} 3\frac{T}{m}\left(1 - \frac{5}{2}\frac{T}{m}\right) & ; \quad \frac{T}{m} \ll 1 \\ \frac{2}{3} & ; \quad \frac{T}{m} \gg 1 \end{cases}$$

From (4.58) it is seen that the critical density increases with temperature. The temperature correction in the non-relativistic limit has been described in Ref. 4. According to (4.59) one gets a limitation of the electrostatic group velocity at the value $\sqrt{2/3}$.

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Notation

Symbols:

$\underline{E}$	: Electric field
$\underline{B}$	: Magnetic field
$\underline{V}$	: Velocity
n	: Density
$f(\underline{x},\underline{v},t)\, d^3\underline{x}\, d^3\underline{v}$	: Number of electrons at the phase space point $(\underline{x},\underline{v})$
λ_D	: Debye length
λ_{De}	: Electron Debye length
L	: Scale length
k	: Wave number
k_0	: Vacuum wave number
λ	: Wavelength
S	: Thermal velocity
C	: Speed of light
q	: Electronic charge
m	: Electronic mass
ω	: Frequency
ω_p	: Plasma frequency
T	: Temperature

Subscripts:

em	: Electromagnetic
es	: Electrostatic
i	: Ion
0	: Equilibrium
1	: Perturbation
gr	: Group
ph	: Phase
$[\cdots]$	: Jump at point z_0: $[a(z)] = a(z_0+) - a(z_0-)$

Relativistic quantities:

a^μ	: 4 - vector; $\mu \in \{0,1,2,3\}$
a^0	: Time component of a^μ
$\vec{a}$	: Space components of a^μ
u^μ	: 4 - velocity
A^μ	: 4 - potential
$T^{\mu\nu}$	: Energy-momentum tensor
$g^{\mu\nu}$	: Lorentz metric
n	: Particle density
p	: Pressure
ρ	: Mass energy density
$\bar{a}^\mu$	: Equilibrium
$\hat{a}^\mu$	: Perturbation

Subscripts:

em	: Electromagnetic
es	: Electrostatic
i	: Ion
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