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proton radius?

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Abstract

The work is devoted to a consideration of the different determinations of the proton charge radius. It is demonstrated that the results from the elastic electron-proton scattering have to be of a higher uncertainty. A review of the hydrogen Lamb shift measurements and the radius determination from them is also presented.

1 Introduction

The proton is the lightest and simplest stable hadronic system. Investigation of its structure is quite important. The charge radius determined by the charge distribution inside the proton is one of the universal fundamental physical constants, because it is important for a number of very different physical problems. Some recent precise results are collected in Table 1 and also presented in Fig. 1. The proton charge radius is defined here as $R_p = \sqrt{\langle r^2 \rangle}$. One can see that the values are obtained from different branches of physics. We give first a short description of all well known and most ‘popular’ results summarized there¹.

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¹Of course, the Lamb shift result mentioned there is too fresh to be ‘popular’ but it is going to be.

Value	Reference	Method
0.809(11) fm	Stanford, 1963	scattering experiment & empirical fitting
0.862(12) fm	Mainz, 1980	scattering experiment & empirical fitting
0.64(8) fm	Draper, Woloshin, Liu, 1990	lattice QCD in chiral limit
0.88(3) fm	Leinweber, Cohen, 1993	lattice QCD & chiral perturbation theory
0.847(9) fm	Mainz, 1996	dispersion relation fitting
0.890(14) fm	Garching, 1997	hydrogen Lamb shift measurements

Table 1: Proton charge radius

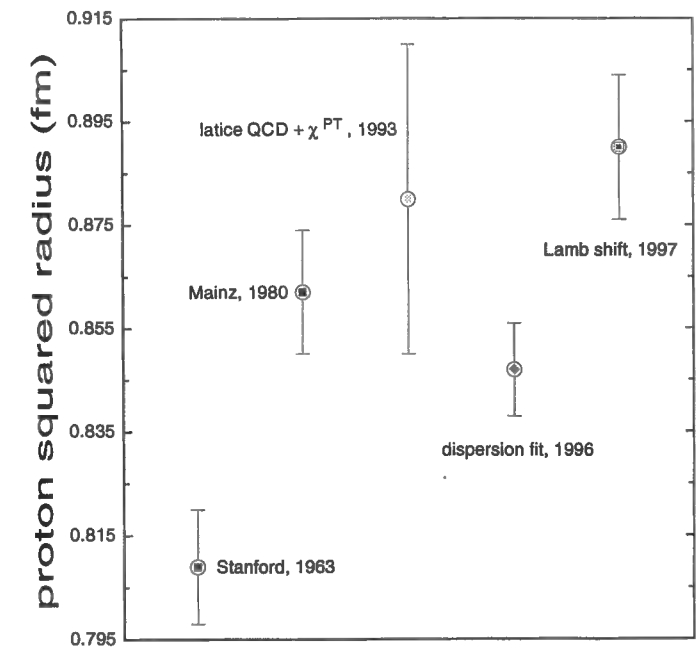


Figure 1: The proton charge radius as it is usually known

- In paper [1] results on electron-proton elastic scattering were obtained and fitted using a simple empirical formula. The experiment has been done by means of the accelerators and particle physics. Theoretically they are based on the particle physics phenomenology and the quantum electrodynamics theory for corrections.
- In work [2] new results for $e-p$ were presented. The radius was found by an extrapolation to zero transfer momentum.
- In paper [3] (see also Ref. [4]) a lattice calculation was performed within the chiral limit, where the pion is massless ($m_\pi = 0$). The result for the radius was found by fitting of the form factor computed there. The lattice calculation are result of the quantum field theory without using of the perturbative expansion in the Euclidian space. As far as the space is not continuous there are a lot of problems there due to a formal description of relations between the physics in the infinite continuous Minkowsky space and in the discrete finite Euclidian one.
- In article [5] a chiral perturbation correction to previous calculations was presented. The chiral perturbation theory is another branch of the quantum field theory where the smallness of the masses of the u and d quarks is used.
- In Ref. [6] a many parameters fit was performed. It included transferred momentum within a wide range. At least two new experimental subfields are involved into this approach. First the wider range of data of the proton electron scattering includes a lot of absolutely different experiments. And second the dispersion approach based on the theory of the analytic properties of the scattering amplitude and other values involves a number of relation between the data from other kinds of collision. For instance, the neutron data obtained from scattering on nuclei are included in the evaluation.
- The measurement of the $1s - 2s$ transition frequency in the hydrogen atom [7] gives a result for the ground state Lamb shift and hence for the proton radius. The precise investigation of the hydrogen spectra involves laser spectroscopy to measure the transition frequencies and quantum mechanics and quantum electrodynamics to calculate all contributions to the energy levels beside the radius term.

These short explanations of the results demonstrate the universality of the proton charge radius. This gives us the possibility to perform some cross-checking of a number of different pieces of our knowledge on the atoms and particles. There are also a number of other subfields where the knowledge of the radius is interesting, but they cannot give any precise result. An example of those can be some models of the proton or neutron. As

some important applications the proton polarizability and the empiric determination of the neutron magnetic radius can be mentioned.

The results are given in Table 1 and Fig.1 within the chronological order, but we will consider them in detail starting with the Lamb shift measurement. Next we will discuss the lattice calculations and then the scattering data and their different fitting. As one of the applications the hydrogen hyperfine structure and so called polarizability contribution is considered in Appendix.

2 The hydrogen Lamb shift for the proton radius

The nuclear dependent correction to the Lamb shift in the hydrogen atom as it is well known is of the form²

$$\Delta E(nl) = \frac{2}{3} \frac{(Z\alpha)^4}{n^3} m^3 R_p^2 \delta_{l0} .$$

If the energy were measured with high accuracy and all other contributions to energy were known one could extract the proton charge radius from hydrogen spectroscopic data. However, it is only possible to measure some differences of the state energies: either splittings or transition frequencies. They are dimensional values and to determine them one has to use some other dimensional reference values. We give a review of all experimental results, but before that we would like to attract the attention to a problem appeared recently due to a significant progress in the optical measurements.

There are two kinds of the optical measurements from which one can extract the value of the Lamb shift. First let us consider absolute optical measurements. They have been done for the $1s - 2s$, $2s - 8s$ and $2s - 8d$ transitions. One problem to utilize their result is unknowledge of the Rydberg constant which mainly determines the value of the transition frequency. The Lamb shift may be found only by using experimental results for two or more transition and solving a system of equations where the constant is one of the variable to find. In this way one gets the value of the Lamb shift from some special difference of two frequencies where the Rydberg constant contribution is canceled:

$$\left(\frac{1}{2^2} - \frac{1}{8^2} \right) \cdot E(1s - 2s) - \left(\frac{1}{1^2} - \frac{1}{2^2} \right) \cdot E(2s - 8s/d) . \quad (1)$$

The other method of optical measurements (so called ‘ground state Lamb shift measurements’) is based on producing difference like in eq.(1) experimentally as a beat frequency. Measurements of $1s - 2s/2s - 4s$, $1s - 2s/2s - 4p$, $1s - 2s/2s - 4d$, $1s - 3s/2s - 6s$ and $1s - 3s/2s - 6d$ transitions have been done.

One can see that in the optical measurements it is not possible to measure the Lamb shift of one separated level. Usually they can determine something like

$$\Delta E_L(1s) - C \Delta E_L(2s) + \dots ,$$

where the known constant C depends on the experiment and usually it lies between 4 and 5. The points stands for the Lamb shift of higher s levels and of levels with higher orbital momentum $l0$. All s level energy depend on the proton radius.

A way of evaluating the data in which a combination of Lamb shifts of different s -states is involved has been considered in our works [8, 9]. It is possible to recalculate all ns shifts to the ground state value by using an auxiliary difference

$$\Delta(n) = \Delta E_L(1s) - n^3 \Delta E_L(ns) ,$$

which is radius-independent. The details of this calculation can be found in our paper [10].

The higher l levels are not radius-dependent on this level of accuracy and they can be also calculated with good enough precision [8, 9].

2.1 Status of the Lamb shift theory

We now can compare the status of the ground state Lamb shift theory and the $\Delta(n)$ theory. All recent results for the $1s$ (or $2s$) Lamb shift are presented in Table 2. In the Table we have included only results obtained directly for the hydrogen atom. However, it has to be mentioned that some works for hydrogen-like ions with

Order	Reference
$\alpha(Z\alpha)^6 m$	Pachucki, 1993, [11]
$\alpha(Z\alpha)^7 m \log Z\alpha$	Karshenboim, 1994, [8]
$\alpha^2(Z\alpha)^6 m \log^3 Z\alpha$	Karshenboim, 1993, [12]
$\alpha^2(Z\alpha)^5 m$	Pachucki, 1994, [13]
	Eides, Shelyuto, Grotch, 1995, [14]
$(Z\alpha)^6 m^2 / M \log Z\alpha$	Doncheski, Grotch, Erickson, 1991, [15]
	Khriplovich, Milshtein, Yelkhovsky, 1992, [16]
	Fell, Khriplovich, Milshtein, Yelkhovsky, 1993, [17]
	Pachucki, Grotch, 1995, [18]
$(Z\alpha)^6 m^2 / M$	Pachucki, Grotch, 1995, [18]
	Yelkhovsky, 1996, [19]
	Eides, Grotch, 1997, [20]
	Yelkhovsky, 1997, [21]
$\alpha(Z\alpha)^5 m^2 / M$	Bhatt, Grotch, 1987, [22]
	Pachucki, 1995, [23]
	Eides, Grotch, 1995, [24]

Table 2: New results for the 1s and 2s Lamb shift

Order	Reference	Contribution
$(Z\alpha)^6 m^2/M$	Pachucki, Grotch, 1995	-7.4 kHz
	Artemyev, Shabaev, Yerokhin, 1995	-7.1(9) kHz
	Yelkhovsky, 1996	2.8 kHz
	Eides, Grotch, 1997	-7.4 kHz
	Yelkhovsky, 1997	-16.5 kHz
$\alpha(Z\alpha)^5 m^2/M$	Bhatt, Grotch, 1987	-20.3 kHz
	Pachucki, 1995	-13.9 kHz
	Eides, Grotch, 1995	-20.7 kHz

Table 3: Theoretical discrepancies for the 1s Lamb shift

the nuclear charge Z can be useful also for $Z = 1$. Particularly, some important numerical results were obtained in works [25] (αm , all order of $(Z\alpha)$, $Z = 5, 10, \dots$) and [26] (m^2/M , all order of $(Z\alpha)$, $Z = 1, 2, \dots$).

One can see from Table 2 that some corrections were investigated by a number of authors and it has to be mentioned that the results were not always in agreement. For instance it was a long and dramatic history of the calculation of the logarithmic corrections $(Z\alpha)^6 m^2/M \log Z\alpha$ which after all were found to be equal to zero [17]. Two actual contradictions of the ground state Lamb shift are presented in Table 3. For the interpretation of the numerical results of Ref. [26], which include also all higher order corrections, it is necessary to remember that uncertainty from the higher order corrections can be estimated as the $(Z\alpha)^5 m^2/M$ term multiplied by $(Z\alpha)^2 \log Z\alpha$, or on level of 1 kHz.

Now we can consider the status of the specific difference $\Delta(n) = \Delta E_L(1s) - n^3 \Delta E_L(ns)$ and the Lamb shift of the p states. One can see from Table 4, that the theoretical status of $\Delta(n)$ and p state energy levels is physically more clear than the status of the ground state Lamb shift. Some details can be found in Refs. [8, 9, 10, 27, 28].

The general expression of the difference $\Delta(n)$ is of the form [10]

$$\Delta(n) = \frac{\alpha(Z\alpha)^4 m_r^3}{\pi m^2} \times \left\{ -\frac{4}{3} \log \frac{k_0(1s)}{k_0(ns)} \left(1 + Z \frac{m}{M}\right)^2 + \right. \\ \left. (Z\alpha)^2 \times \left[\left(4(\log n - \psi(n+1) + \psi(2)) - \frac{77(n^2-1)}{45n^2}\right) \log \frac{1}{(Z\alpha)^2} + A_{60}^{VP}(n) + G_n^{SE}(Z\alpha) \right] \right. \\ \left. - \frac{14}{3} \frac{Zm}{M} \left(\psi(n+1) - \psi(2) - \log n + \frac{n-1}{2n} \right) \right\} + \frac{\alpha^2(Z\alpha)^6 m}{\pi^2} \log^2 \frac{1}{(Z\alpha)^2} B_{62},$$

where $\log k_0(ns)$ is the Bethe logarithm and $\psi(z) = (d/dz) \log \Gamma(z)$. Here:

- $G_n^{SE}(Z\alpha)$ is the *one-loop self-energy* correction of order $\alpha(Z\alpha)^6 m$ and higher;
- $A_{60}^{VP}(n)$ is the $\alpha(Z\alpha)^6 m$ -contribution of the *vacuum polarization* [29];
- B_{62} is the *leading logarithmic two-loop* correction coefficient [8, 30, 27].

The final results for the difference $\Delta(n)$ in the hydrogen and deuterium atoms as well as for their isotopic shift were found in Ref. [10] and they are summarized in Table 5.

²We use a convention: $\hbar = c = 1$, $\alpha = e^2$. The nuclear charge Z is useful to classify the QED contributions and in the hydrogen atom indeed one has $Z = 1$. We always give expressions for E as for the energy, but the numerical values (in kHz) are given for the corresponding frequency value $f = E/\hbar = E/2\pi$.

Order	$E_L(1s)$	$\Delta(n)$	$E_L(np)$
$\alpha(Z\alpha)^6 m$ $\alpha(Z\alpha)^8 m \log^3 Z\alpha$	coefficient ≈ 30 unknown	coefficient ≈ 1 0	coefficient ≈ -1 0
$\alpha^2(Z\alpha)^6 m \log^3 Z\alpha$ $\alpha^2(Z\alpha)^6 m \log^2 Z\alpha$	30 kHz unknown	0 known	0 known
$\alpha^3(Z\alpha)^4 m$	unknown	0	known
$(Z\alpha)^6 m^2 / M$ $\alpha(Z\alpha)^5 m^2 / M$	disagreement disagreement	0 0	agreement 0
$(Z\alpha)^4 m (mR_p)^2$	radius-dependence	0	0

Table 4: Comparison of status of $E_L(1s)$ and $\Delta(n)$

n	$\Delta^{Hyd}(n)$ [kHz]	$\Delta^{Deu}(n)$ [kHz]	$\Delta^{Iso}(n)$ [kHz]
2	-187232(5.5)	-187225(5.5)	7.3
3	-235079(10)	-235073(10)	5.9
4	-254428(12)	-254423(12)	4.7
5	-264162(15)	-264158(15)	4.0
6	-269747(15)	-269743(15)	3.5
7	-273246(16)	-273243(16)	3.2
8	-275583(17)	-275580(17)	3.0
9	-277221(18)	-277218(18)	2.9
10	-278413(19)	-278410(19)	2.7
11	-279308(19)	-279305(19)	2.6
12	-279996(20)	-269993(20)	2.5

Table 5: Results of $\Delta(n)$ for $n = 2 \dots 12$

By using the differences considered above one can easily recalculate the Lamb shift of the $1s$ state to the Lamb shift of the $2s$. In this work we prefer to discuss the Lamb splitting between the $2s_{1/2}$ and $2p_{1/2}$ states. Our motivation to do that is the possibility to compare the data evaluated with $\Delta(n)$ (the optical measurement) to the one which obtained directly (the radiofrequency measurement).

Above we have compared the status of the $2s$ Lamb shift and the difference $\Delta(n)$ and given the complete theoretical expression for the second [10]. The theory of the $2s$ Lamb shift is much more complicated. We mainly agree with a consideration in review [31]. However, it has to be mentioned that compared to it, our result is shifted by -3.6 kHz because the $\alpha^2(Z\alpha)^6 m \log^3 Z\alpha$ -correction [12] has not been included there. We would also like to explicitly present here the sources of the theoretical uncertainty:

- unknown $\alpha(Z\alpha)^7 m$ and higher order corrections are estimated to 1 kHz;
- $\alpha^2(Z\alpha)^6 m \log^2 Z\alpha$ and higher order terms can give up to 2 kHz;
- the $\alpha^3(Z\alpha)^5 m$ contributions previously estimated here to 2 kHz need more understanding.

2.2 Status of the Lamb shift experiment

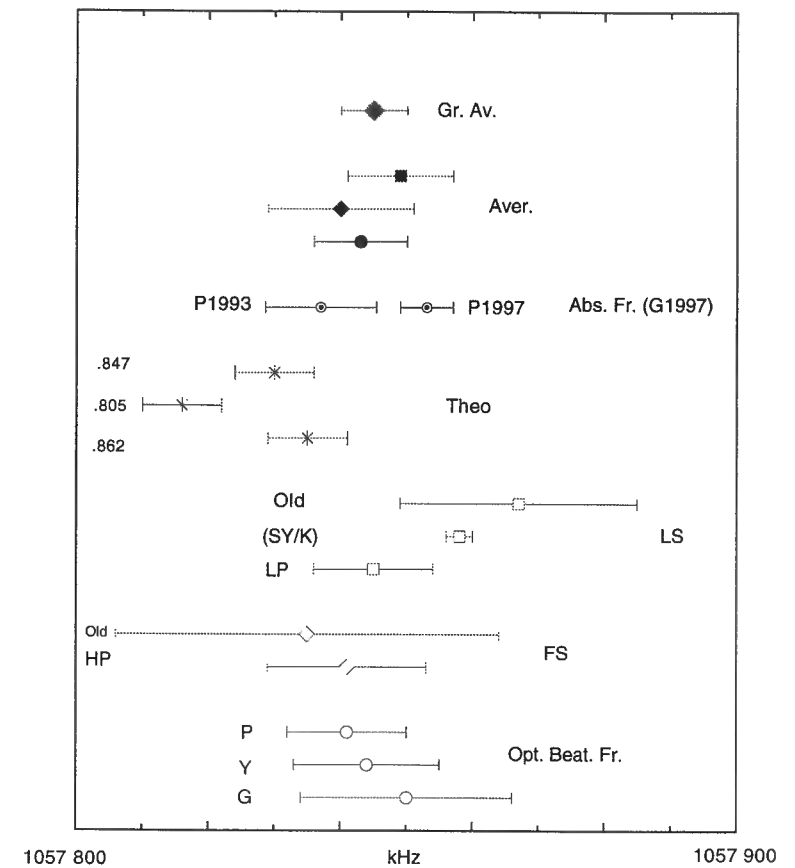


Figure 2: Different values of the Lamb splitting $E(2s_{1/2}) - E(2p_{1/2})$

All important result for the Lamb splitting ($2s_{1/2} - 2p_{1/2}$) are presented in Fig. 2. It is helpful first to describe all labels in the text and next to discuss them. The references (and the transitions) of the experiments and theory presented in the picture are:

Opt. Beat. Fr. **Optical beat frequency measurements** were realized with a high accuracy only recently:

G Measurement of $1s - 2s/2s - 4s$, Garching, 1994: [32].

Y Measurement of $1s - 2s/2s - 4p$, Yale, 1993: [33].

P Measurement of $1s - 2s/2s - 4p$, Paris, 1996: [34].

FS Fine structure measurements deal with the radiofrequency splitting ($2p_{3/2} - 2s_{1/2}$).

HP The best and most recent result is of *E. W. Hagley and F. M. Pipkin, 1994* [35].

Old Older experiments were not very precise and it is more convenient to consider them altogether. They include results obtained by

- *B. L. Cosens and T. V. Vorburger, 1969* [36];
- *T. W. Shyn et al., 1971* [37];
- *K. A. Safinya et al., 1980* [38] (result corrected according to *E. W. Hagley and F. M. Pipkin, 1994* [35]).

LS Lamb splitting measurements have been performed directly for the classical splitting $2s_{1/2} - 2p_{1/2}$.

LP The result by *S. R. Lundeen and F. M. Pipkin, 1981* [39] is one of two relatively recent ones.

SY/K The other experiment was performed by Sokolov and Yakovlev and the result corrected by us. The experiment properly by *Yu. L. Sokolov and V. P. Yakovlev, 1982* [40] gives a result for the ratio of the $2p_{1/2}$ decay rate and the splitting. The correction to decay rate was found in Ref. [8, 41] by *S. G. Karshenboim, 1994*. See also *Comments to On the Accuracy of Lamb Shift Measurements in Hydrogen, (Physica Scripta, 55 (1997) 33–40)* by *V. G. Pal'chikov, Yu. L. Sokolov, and V. P. Yakovlev* [42].

Old Older experiments are summarized in one point. The results taken into account include values obtained by:

- *D. A. Andrews and G. Newton, 1976* [43];
- *R. T. Robiscoe and T. W. Shyn, 1970* [44].

Theo Some **theoretical** values are presented too. Different results appear due to the different values of the proton radius discussed above. They are labeled with the proper radius values (see Table 1):

- .862 – Mainz scattering result [2];
- .805 – Stanford scattering result [1];
- .847 – Mainz dispersion fitting result [6].

Abs.Fr.(G1997) The **comparison of absolute optical frequencies** is presented separately from other experimental points because in contrast to them it contains data from *two* independent experiments. The other reason is that this gives now the results with the highest accuracy (except the Sokolov–Yakovlev result which is discussed below). The $1s - 2s$ measurement [7] done in Garching (1997) cannot be used alone. The other absolute measured value used to find the Lamb shift is a Paris result for the $2s - 8s/8d$ frequency. We give two points due to two different Paris experiment:

P1993 Earlier result: Paris–1993 [45],

P1997 Recent result: Paris–1997 [46].

Aver It is also convenient to compare results of *Abs.Fr.(G1997)* with some **average values**. We give average values for three kind of experiments. The labels are easily to recognize. We would like to comment them:

OBF The evaluation is performed with by $\Delta(n)$ [10].

FS The result is obtained without using $\Delta(n)$, but with applying the theoretical results for the fine structure interval $2p_{1/2} - 2p_{3/2}$ [28].

LS The result is obtained without using either $\Delta(n)$ or the fine structure $2p_{1/2} - 2p_{3/2}$. The SY/K-value is not taken into account here.

Gr.Av. The **grand average value** is found over the three average values mentioned above.

Before a detailed discussion we would like to attract the attention to an **important point**. As it is mentioned the difference $\Delta(n)$ has been involved into evaluation of optical beat frequency (*Opt.Beat.Fr*) and the absolute frequency (*Abs.Fr.*) measurements the data. The fine structure (*FS*) and the Lamb splitting (*LS*) measurements data are needed no results of $\Delta(n)$. One can see in Fig. 2 that these two group of the results ($\Delta(n)$ -dependent and $\Delta(n)$ -independent) are in fair agreement.

Now we are listing some *problems for the hydrogen measurements adjustment* needed to be solved for extracting the proton radius from the Lamb shift data.

- The first problem to solve is to fix the experiments to be included and the uncertainties to be used for them.
 - Sokolov and Yakovlev experiment [40] and its uncertainty is an open question because some criticism of Hinds [47];
 - Older experiments altogether can give an uncertainty of about 10 kHz. Not all of them are taken into account in Fig. 2. We would like here to mention especially some Lamb splitting experiments and one for the fine structure.
 - The value of the Lamb splitting [36] is included and the uncertainty ($\delta = 42 \text{ kHz}$) is taken by us according to the original work in contrast to a consideration by *B. N. Taylor et al., 1969* [48] (the uncertainty there is $\delta = 64 \text{ kHz}$).
 - The result of the Lamb splitting by *S. Triebwasser et al., 1953* [49] is excluded because the work includes results for two lines. Being corrected (see [48] for details) they are inconsistent; $\delta = 64 \text{ kHz}$ (according *B. N. Taylor et al.* [48]).
 - The experiment by *Yu. L. Sokolov, 1973* [50] for the Lamb splitting is not included being in straight disagreement with all other measurements ($\delta = 40 \text{ kHz}$).
 - The result for the fine structure by *S. L. Kaufman et al., 1971* [51] is excluded also being in straight disagreement with all other measurements ($\delta = 40 \text{ kHz}$).
- One can see from the picture that the two absolute measurement of the $2s - 8d/8s$ transition from Paris are in a disagreement: the earlier result of 1993 [45] is

$$f(2s - 8d_{5/2}) = 770\,649\,561\,567(10) \text{ kHz}$$

and the recent result of 1997: [46] is

$$f(2s - 8d_{5/2}) = 770\,649\,561\,585(5) \text{ kHz}.$$

Actually these are two different experiment where different standards have been used and the results are expected to be partly independent. In another work [52] the older experiment result is presented as

$$f(2s - 8d_{5/2}) = 770\,649\,561\,571(12) \text{ kHz}.$$

It is not clear if we have to use one of them (which one ?) or to find some average value.

- It has to be also mentioned that actually there are a number of older less precise measurements from Paris which are quite correlated.
- Another question that appears for such high accuracy as 3 kHz for the $2s$ Lamb shift is the fine structure constant to be used. The different choice of it can lead to shift of the $E(2s_{1/2}) - E(2p_{1/2})$ splitting value up to 1 kHz for the results obtained by the *optical beat frequency* and *absolute frequency* measurements. For the $E(2s_{1/2}) - E(2p_{1/2})$ value from the *fine structure* measurements the shift is up to -2 kHz, where ‘-’ indicates that the shift is in the opposite direction to the one for the results extracted from the *Opt.Beat.Fr.* and *Abs.Fr.* experiments. The results for the direct measurements of *Lamb splitting* cannot be significantly shifted. That is because all changes due to correcting of the fine structure constant value come via the $(Z\alpha)^4 m$ term (the relativistic correction to the Schrödinger energy which is also responsible for the hydrogen fine structure). This term is about an order of magnitude **larger** than the Lamb shift properly. Due to these different signs the *grand average* value is less sensitive to the choice of α , but according to this choice the set of results may be more or less inconsistent.
- One can combine the data and extract the value of the fine structure constant. The uncertainty is of the level of $3 \cdot 10^{-7}$ and it seems that it cannot be reduced in the nearest future.

Some new problem can appear when taking into account *the deuterium results*. It is important to remember:

- Isotopic shift of the $1s - 2s$ transition frequency has been measured in Garching within **0.15** kHz uncertainty [53].
- Isotopic shift of $\Delta(n)$ with uncertainty less than **0.05** kHz has been obtained by *S. G. Karshenboim, 1997* [10] (see Table 5).

- That gives the isotopic shift
 - of the $1S$ Lamb shift with an uncertainty of **0.17** kHz;
 - of the $2S$ Lamb shift with an uncertainty of **0.02** kHz.
- The list of the deuterium results similar to the case of the hydrogen has to include a number of old and recent ones. They are shortly discussed below³.
 - The measurement of the Lamb splitting by *B. L. Cosens, 1968* [54] has been correlated with the hydrogen one ($\delta = 33 \text{ kHz}$).
 - The experiment by *S. Triebwasser et al., 1953* [49] for the Lamb splitting also has contained correlations with the hydrogen result and led to the same problem as for hydrogen: the results for two lines corrected according to *B. N. Taylor et al., 1969* [48] are inconsistent ($\delta = 35 \text{ kHz}$).
 - The work by *E. S. Dayhoff et al., 1953* [55] devoted to the fine structure have contained a disagreement between the results for two lines ($\delta = 53 \text{ kHz}$).
 - The optical beat frequency measurement of $1s-2s/2s-4s$ transitions in the deuterium atom (Garching, 1994, [32]) has been done with the same experimental setup as for the hydrogen ($\delta = 28 \text{ kHz}$).
 - The best absolute measurement has been performed recently for the $2s-8s/8d$ transitions in Ref. [46] (Paris, 1997) and it has been correlated with the hydrogen measurement; the uncertainty is the same as for the hydrogen ($\delta = 5 \text{ kHz}$).
- One can see that **all** deuterium results are more or less correlated with the hydrogen results obtained by the same teams.
- It seems that the Sokolov and Yakovlev experiment is the only direct way where it could be possible to obtain uncertainty below 1 kHz for the 2s Lamb shift. Due to a perfect reproducibility, the unknown systematic error can be expected to be the same for hydrogen and deuterium.

3 Lattice calculations

One of the other methods that gives a value for the proton radius (see Table 1 and Fig. 1) is the approach based on the lattice calculation within the chiral limit in which the pion mass is equal to zero. We expect that the result of Ref. [5] included in the Table 1 and Fig. 1 is not quite correct. Our opinion is based on the following critical remarks:

- Small lattice ($24 \times 12 \times 12 \times 24$) has been used in the calculation. Some progress is possible: in a recent evaluation of the neutron form factor in Ref. [56] a lattice $20^3 \times 32$ was used. In case of the small lattice only a not too large number of discrete momenta are available. The proton has to occupy a relatively large portion of the volume of the whole space.
- That is not an *ab initio* calculation. It includes a number of free auxiliary parameters found by a barions masses fitting.
- The chiral limit ($m_\pi = 0$) also leads to some limitation for the available values of the low momentum.
- The finite space implies neither $q^2 = 0$ nor $q^2 \rightarrow 0$, because only some discrete value of q are allowed. The radius cannot be found directly, but only by some fitting.
- As far as the form factor $G_E(q^2)$ could be found only at not too small q^2 (e.g. $0.16 \text{ GeV}^2 \simeq 4 \text{ fm}^{-2}$), it has been fitted with monopole or dipole fits. They have obtained results for only a few values of q^2 and used a normalization at $q^2 = 0$.
- The result looks very sensitive to chiral perturbing: the value of the proton radius was shifted from **0.64(8)** fm [3] to **0.88(3)** fm [5]. It may seem quite strange that using the chiral correction a smaller uncertainty has been obtained.
- *But a more important question is the status of their estimation of the uncertainty.*

³The items marked by ‘–’ should rather be excluded from adjustment in contrast to ones with ‘◦’ which should be taken into account.

- Due to the simple Yukawa picture for the proton with the charged pion clouds it is not quite clear how the proton charge radius can be finite in the massless pion limit. The virtual neutral pions also lead to a divergency which is similar to the well known divergency in the electric form factor of the electron in the usual quantum electrodynamics.
- The proton radius is also expected to be infinite in the pure chiral limit according to the correcting fit of Ref. [5].
- But in the previous work [3] the same authors just used only the chiral limit and obtained a finite result !
- If they are not able to recognize an actual divergency how one can believe their result and especially the uncertainty ?
- We expect that was possible maybe because they used only a few of the values of not too low momenta q^2 and the dipole or monopole fits. The divergency were actually cut off with the finite volume of the space. It also has to be mentioned that the normalization $G(0) = 1$ was also applied by them, but it has no clear physical meaning because in finite space $q^2 \neq 0$.
- The results are actually sensitive to the fitting procedure, which can allow to obtain a finite value instead of infinity.

4 Scattering data

We now can start to discuss the other traditional way to investigate the proton radius based on the unpolarized elastic electron-proton scattering. The cross-section described by the well known Rosenbluth formula is of the most simple form in terms of the Sachs form factors:

$$\frac{d\sigma(E_0, \theta)}{d\Omega} = \left[\frac{d\sigma(E_0, \theta)}{d\Omega} \right]_0 \times \left[A(q^2) G_E^2(q^2) - \frac{q^2}{4M^2} B(q^2, \theta) G_M^2(q^2) \right], \quad (2)$$

where

$$\left[\frac{d\sigma(E_0, \theta)}{d\Omega} \right]_0 = \frac{\alpha^2}{4E_0^2} \frac{\cos^2 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} \frac{1}{1 + \frac{2E_0}{M} \sin^2 \frac{\theta}{2}}$$

is the cross-section for scattering with a spinless point-like particle (instead of the proton) and $q^2 = q_0^2 - \mathbf{q}^2 < 0$.

The coefficients in eq.(2) are known

$$A(q^2) = \frac{1}{1 - q^2/4M^2}$$

and

$$B(q^2, \theta) = \left(\frac{1}{1 - q^2/4M^2} + 2 \tan^2 \frac{\theta}{2} \right).$$

In the limit of the small momentum q^2 they are equal to

$$A(0) = 1,$$

$$B(0, \theta) = 1 + 2 \tan^2 \frac{\theta}{2}.$$

The proton (electric) squared radius in terms of the form factor is defined as

$$R_p^2 = \frac{1}{6 G_E(0)} \frac{\partial G_E(q^2)}{\partial q^2} \bigg|_{q^2=0}. \quad (3)$$

The Sachs form factors G_E and G_M are not only useful in the particle physics. The others for particle with spin 1/2 are the Dirac (F_1) and the Pauli (F_2) form factors. In their terms the vertex of the emission of a virtual photon by a real proton is of the form

$$\Gamma_\mu = F_1(q^2) \gamma_\mu + F_2(q^2) \left(-\sigma_{\mu\nu} \frac{q^\nu}{2M} \right).$$

There are two relations between the different form factors:

$$F_1(q^2) = \frac{G_E(q^2) - (q^2/4M^2)G_M(q^2)}{1 - q^2/4M^2}$$

and

$$F_2(q^2) = -\frac{G_E(q^2) - G_M(q^2)}{1 - q^2/4M^2}.$$

By measuring the scattering cross-section and applying the Rosenbluth formula one can obtain the values of the form factors. A number of problems appear when **fitting the scattering data**.

- *Dipole approximation.*

The so called dipole approximation is of the form

$$G_E(q^2) = \frac{G_M(q^2)}{1 + \kappa} = \left[\frac{\Lambda^2}{\Lambda^2 - q^2} \right]^2.$$

This is a very good approximation for any preliminary estimation, but it has no theoretical status. It was used in some old fitting.

- *Extrapolation.*

In the Mainz experiment [2] they used a three term extrapolation

$$G_E(q^2) = \frac{G_M(q^2)}{1 + \kappa} = a_0 + a_1 q^2 + a_2 q^4.$$

The same was done in some older works too (see e. g. [61]). From the charge normalization one could expect $a_0 = 1$. But actually the problem is not so simple and we discuss that below after considering the problem due to the magnetic form factor.

- *Magnetic radius.*

In the Mainz-1980 evaluation of the data as well as in a lot of previous works it a relation

$$G_E(q^2) = \frac{G_M(q^2)}{1 + \kappa}$$

was used for fitting. That definitely works for $q^2 = 0$

$$G_E(0) = \frac{G_M(0)}{1 + \kappa} = 1. \quad (4)$$

and definitely does not work in general because, e. g.

$$G_E(4M^2) = G_M(4M^2).$$

Hence, this is only an approximation and it is not quite clear what systematic errors come from the difference between $G_E(q^2)$ and $G_M(q^2)/(1 + \kappa)$. In the Mainz experimental work [2] they used for the evaluation only data for which the magnetic effects are less than 10%. That means that the uncertainty for R_M cannot be smaller than 10 times δR_E , if we have nothing else to estimate the difference $R_E - R_M$. In the Stanford work they used also some higher momentum measurements. We give all accurate experimental points for the electric form factor in two pictures. In Fig. 3b results are included for range of momentum used in the Stanford fitting (up to $q^2 = 3 \text{ fm}^{-2}$), but in Fig. 3a we keep only the same momenta as in the Mainz experimental fitting for which magnetic effects can be neglected ($q^2 \leq 1.4 \text{ fm}^{-2}$). **Experimental status of the problem** can be illustrated with a result by *H. Treissen and W. Schütz, 1974* [57] done for the not too high momenta q^2

$$G_E(q^2) = 1.01(3) \cdot \frac{G_M(q^2)}{1 + \kappa}. \quad (5)$$

In case of a low momentum transfer one can write

$$G_E(q^2) \approx 1 + \frac{R_E^2 q^2}{6} \quad (6)$$

and

$$G_M(q^2) \approx (1 + \kappa) \left(1 + \frac{R_M^2 q^2}{6} \right), \quad (7)$$

and for all low energy of the Mainz data that they used for the fitting it was

$$1.5 \cdot 10^{-2} < (G_E - 1) \approx \frac{R_M^2 q^2}{6} < 15 \cdot 10^{-2}.$$

That means that for so low momenta q^2 the result of eq.(5) comes almost from the normalization eq.(4) and actually cannot give any information on the radii.

For high q^2 there are no direct connection between the radii and form factors, but for the low momenta as we can see the electric radius term is of the order 10^{-2} . Thus there is no significantly accurate result for critical comparison of R_E and R_M .

- *Normalization of cross section.*

Measuring cross section one has to take care of a proper normalization factor. Due to the experimental normalization the result of extrapolation

$$G_E(q^2) = a_0 + a_1 q^2 + a_2 q^4. \quad (8)$$

to $q^2 \rightarrow 0$ has **not** to give a_0 equal to 1 exactly. That is not possible due to the measurement nature of the normalization. But of course a_0 has to be in agreement with 1 **within the experimental uncertainty**. And an more point: the uncertainty mentioned has to be interpreted as the systematic error and thus the value of a_0 can be different for different experiments. E. g. in the Mainz-1980 work they used different values to fit the data of three experiments:

- $a_0(\text{Mainz}) = 1.0014$ [2];
- $a_0(\text{Orsay}) = 1.020$ [60];
- $a_0(\text{Saskatoon}) = 1.008$ [61].

The physical meaning of the constant a_0 is: *the function $G(q^2)$ determined straightforwardly from the Rosenbluth formula is **not** the true form factor*. The true one is $G(q^2)/a_0$. The true value of radius has to be defined as

$$R_p^2 = \frac{a_1}{6 a_0}.$$

From the Mainz experiment article [2] it is not quite clear if they used the definition above or

$$R_p^2 = \frac{a_1}{6}.$$

The last equation is wrong. That can be important for incorporation of the Orsay and Saskatoon data. In the dispersion fitting work [6] they have not used different normalizations for different experiments, but only

$$G_E(0) = 1.$$

As a result they have underestimated the uncertainty because all systematic errors are divided there by a large statistical factor due to the number of the experimental points.

We would like to discuss here shortly a well known contradiction between the Stanford [1] and Mainz [2] empirical fitting results for the radius. The Mainz result was obtained by evaluating Mainz data properly [2], Saskatoon data [61] and Orsay data [60]. The Stanford radius was found by treating the Stanford [1], Orsay [60] and some less precise data. The last are not presented in Figs. 3. One can easily see from the Fig. 3b, that the data used in the Mainz work are much anymore representative and significantly more precise. Some high momentum points in Fig. 3a cannot really change the situation because they are not too accurate. We will not consider more the Stanford data here.

An other important point is due to the **QED corrections**. In order to determine the true proton form factor from the Rosenbluth formula one has to take into account a number of such corrections:

- The main radiative corrections are due to the electron form factor and the bremsstrahlung, and the electronic vacuum polarization. The next important term is the two photon exchange. The complete result in the one loop approximation was found in Ref. [64]. However, in earlier experimental works some incomplete results (see Refs. [62, 63]) were used. E. g. the QED corrections according to Ref. [63] were used in papers [1, 60], but in the evaluation in work [61] the results of Ref. [62] were used. Before including the data mentioned above to any compilation it is necessary to correct them according to Ref. [64]. It is not clear if that has been done or not in the both Mainz compilations.
- Two-loop radiative corrections of the order α^2 include large squared logarithms $\log(q/m_e)$. In Ref. [64] which is now used as the most complete QED result the authors claimed that part of the two-loop corrections may be taken into account using an exponential of the one-loop correcting factor δ_1

$$1 + \delta_1 \rightarrow \exp(\delta_1) .$$

That may allow to include some leading logarithmic terms. In earlier experimental works they were not taken into consideration and it is not clear if they have been actually included in the Mainz experiment evaluation [2]. In the Mainz-1980 work there is no prescription how the QED corrections have been treated. The two-loop correction may contribute on the 1% level of the $(G - 1)$ value for a few lowest momenta q^2 of the Mainz experimental data. The result for the two-loop vacuum polarization is well known and the two-loop form factor of the electron was found in Ref. [59].

- The muonic and hadronic vacuum polarization effects can be also important. The muonic and hadronic vacuum polarization leads to a correction lying in a range from 1% to 0.5% of the $G - 1$ value for all experimental points in Ref. [2]. It has to be compared with about the 1% precision in the radius determination.
- In Mainz-1980 there are no explicit equations for the Rosenbluth scattering. In earlier works often the massless electron approximation $m_e = 0$ was used. The correction is of the order m_e^2/E_0^2 and it has to be $10^{-4} - 10^{-5}$. It is not clear if older data are corrected and if in Mainz-1980 data this approximation has been used or not. Being E_0 -dependent it cannot be incorporated in the fit.

Thus our **QED summary** contains two statement

- It is not clear if the older data from Refs. [60, 61] are included in the final Mainz compilations being corrected according to Ref. [64] or not.
- All QED corrections which can be important on the 1% level precision of the charge proton radius determination have been known and they have to be taken into account when fitting the data.

Now we would like to discuss some results from the dispersion fitting approach. It allows to incorporate data with both high and low momentum transfer. The results for electric and the magnetic proton radii as well as for the magnetic neutron radius have uncertainties on the level of 1% or 0.009 fm [6]. That seems strange because:

- The low energy fitting and high energy one can be done absolutely separately.
- It is clear that we have some additional information only for the proton electric radius which comes from the low energy scattering when the magnetic form factor effects can be neglected. How can the uncertainties be the same ?
- The low energy scattering is expected according to the empirical fit of Ref. [2] to lead to **0.862(12)** fm, and one can obtain the high energy part from re-evaluation back from the average value **0.847(9)** fm [6]. It is close to **0.832(12)** fm and that is it straight disagreement with the low energy result [2].

Concerning the second remark on the different uncertainties expected by us, we would like to say that the fitting has been done in Ref. [6] for the isoscalar

$$F_i^S(q^2) = \frac{F_i^p(q^2) + F_i^n(q^2)}{2}$$

and isovector

$$F_i^V(q^2) = \frac{F_i^p(q^2) - F_i^n(q^2)}{2}$$

form factors, combined from the Dirac ($i = 1$) and Pauli ($i = 2$) form factors of the proton and neutron. We expect that the authors have obtained rather results for the fitting parameters but they are not ready to estimate properly the uncertainty for their specific function like the radii.

Due to the different data from the **low and high energy scattering** we first discuss the different physics of the correction.

- The *two-loop QED correction* are of different magnitude for the low and high momentum.
 - The *low energy* scattering is sensitive to them.
 - The *high energy* one is not.
- The *muonic and hadronic vacuum polarization* are of the same relative 1% level for both of them.
 - The *low energy* data are sensitive.
 - The *high energy* ones are sensitive as well.
- The *analytic properties* of the form factors can be of use.
 - For the *low energy* data evaluation they are not needed. It is a pure empirical fitting
 - The *high energy data* treatment in Ref. [6] has been actually based on their using.
- The *normalization of the experimental data* is the of one important points as we show below.
 - The *low energy* data are sensitive to that up to the level of 10^{-4} .
 - * In the Mainz-1980 work [2] it is included as a fixed parameter with the different values for different experiments.
 - * In the Mainz-1996 paper [6] the normalization of the electric form factor is fixed absolutely to be equal to 1 at $q^2 = 0$. In this way one has to take care of the systematic errors due to correlation of the data from the same experiment.
 - In case of the *high energy* scattering, the data are sensitive to the normalization only on level up to 10^{-2} .
- The *magnetic form factor effects* also have to be mentioned.
 - The *low energy* scattering is a mainly pure electric form factor scattering and magnetic effect are small.
 - The *high energy* data include such effects significantly.

The short **summary** of the comparison '**low energy against the high one**' can be described with following listing:

- The *low energy* data are
 - sensitive to everything;
 - + needed no additional theory (e. g. like the dispersion approach);
 - ± only for G_E .
- The *high energy* data are
 - + not so sensitive;
 - needed to be treated with using significantly a knowledge of the analytic properties;
 - ± both for the electric and magnetic form factors.

The items labeled by '+' are advantages and the ones with '–' are disadvantages.

Now after the comparison of the low energy and high energy data we would like to discuss some **interpretation of the double Mainz difference**. We have reported on some possible disagreement between the low energy and high energy data, but actually there are two options:

- The disagreement can be actual an one and it has been discussed some possible corrections which are different for the high and low energy data.

- In the new Mainz work [6] they could obtain the different result from the one in Ref. [2] for the low energy part of the data, because they could use another fitting. Some different details of their approaches have been also discussed above.

Unfortunately in Ref. [6] the authors have given no discussions which part of the data is mainly statistically responsible for the final result of the proton radius. If we suppose that the difference appeared from the low energy data, the main source of them is the experimental results of Ref. [2]. Maybe some clarifying remarks have been just done by Wong (ironically his work [58] was appeared in 1994 before the publication of the paper [6] two years later). In the quoted work, only the Mainz data have been fitted according eq.(8), but in three different ways. We present all of them in Fig. 4.

- The normalization constant is fixed to $a_0 = 1.0014$. The result is $R_p = \mathbf{0.863(9)}$ fm close to $R_p = \mathbf{0.862(12)}$ fm from Mainz experimental paper [2]. The procedure is also close to the one of Ref. [2]. The difference may come from including the data of the two other less precise experiments or from a different treatment of the systematic errors.
- The normalization constant is fixed to $a_0 = 1$. The result is $R_p = \mathbf{0.849(9)}$ fm close to $R_p = \mathbf{0.847(9)}$ fm from the dispersion fitting work [6]. The same normalization was used there.
- The normalization parameter a_0 has been treated as a free parameter to be found from the fit. The results are $a_0 = 1.0028(22)$ and $R_p = \mathbf{0.877(24)}$ fm.

We think that only the last fitting can be correct, because the normalization constant a_0 cannot be measured with sufficient accuracy and has to be determined by fitting. It has to be mentioned here that in Ref. [2] a special run of measurements for the absolute gauging has been performed. The run included 5 points with the systematic errors equal to either 0.44% or 0.46% and the statistical errors in a range between 0.42% and 1.28%. That means that the last fitting is actually in agreement with the experimental gauging within its 0.7%-uncertainty and there is no reason to fix the normalization as it was done in the first two fits (and in Refs. [2, 6]).

The final uncertainty of the last result is significantly larger than of the two first. Such large changes of the uncertainty are due to two simple reasons. The fit is almost a straight line. To determine it one has to know a few points from the both edges of the data. For the straight line the value of the reference interval is quite important for the uncertainty. When a_0 is fixed exactly one extra point at $q^2 = 0$ is actually included. The reference interval is larger, because for the free normalization one has to use actual results of the cross-section measurement and in this case $q_{min}^2 = 0.14 \text{ fm}^{-2}$. So we can see that for the free normalization the reference interval is now smaller and the number of the fit parameters is larger.

5 Conclusions and discussions

We can now summarize some results on the determination of the proton radius. To the values presented in Fig. 1 we have added in Fig. 5 five other values. First we have included without any discussion an old result with a more reasonable uncertainty (that is fitting in Ref. [61] of the Saskatoon data). Next we present in Fig. 5 the Wong fitting result with the free normalization approach. We also give three values based on the hydrogen Lamb shift study. One appears from using of the grand average value, and the others come from the absolute measurements. Due to the discrepancy mentioned above between the two Paris measurements we give the results for both $2s - 8s/d$ transition frequency values.

5.1 What has been actually done

- The Lamb shift value for the proton charge radius significantly depends on the data used. It is necessary to adjust all data. The uncertainty includes 0.010 fm (theory of the 1s Lamb shift), 0.002 fm (theory of $\Delta(2)$), 0.010 fm (expected experimental uncertainty). The final uncertainty is going to be 0.014 fm.
- The lattice calculations cannot give any good result at least because only a few of the points for the form factor are found at moderate momenta. It is not enough for a successful fitting. We also have some doubts due to the uncertainty given in Refs. [3, 5].
- The scattering data include an underestimation of the uncertainties. The best ones are from Mainz, and due to the Wong fitting with the free normalization it seems that the reasonable uncertainty has to be close to 0.024 fm (i. e. twice larger than in the original work [2]).

- It is not clear what is the actual uncertainty of the radius from the Mainz dispersion fitting and which part of data have actually contributed to the result.

5.2 What can be done

- One of the most important values for atomic applications is $R_E^2 - R_M^2$. Maybe it is less dependent on the proton model. That problem has to be investigated. The value is to be helpful for extrapolating the low energy data, for extracting R_M from the Lamb shift measurement and for calculating of the Zemach correction (see Appendix for details).
- Adjustment for the overall hydrogen data has to include also results for the proton radius. But the problem is: all results are correlated because obtained by fitting of the experimental data. The newer compilations of them are wider, but include also the older results.
- Maybe a measurement of the muonic hydrogen Lamb shift can give a more safe value, but that is an open question. In case of the hydrogen-like ion of the helium-4, a high accuracy has been obtained in an experiment [68], but the result has not been able to be reproduced by an other independent team [69]. So the safety can be just a problem.
- It is not actually clear what the true uncertainties of the Zemach correction with the Mainz dispersion fitting are.

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A Hydrogen hyperfine structure and the proton structure

The hyperfine splitting is much more sensitive to the proton structure than the Lamb shift. The main nuclear structure dependent contribution (so-called ‘Zemach correction’) is of the form

$$\Delta\nu(Zemach) = \nu_F \frac{2Z\alpha m_e}{\pi^2} \int \frac{d^3\mathbf{q}}{q^4} \left[\frac{G_E(-\mathbf{q}^2)G_M(-\mathbf{q}^2)}{1+\kappa} - 1 \right].$$

The Zemach correction includes integration of the form factors, but the most important part comes from the low momenta. That is because the high momenta asymptotic behavior of the integral is determined by ‘-1’ and it is form factor independent. The low momenta asymptotics are defined by eqs. (6) and (7) and the result for the Zemach integral is proportional to $R_E^2 + R_M^2$. So, the result depends straightly on the both proton radius. The low momentum part of the integral comes from the momentum below $0.35 \text{ GeV} \simeq 3 \text{ fm}^{-2}$ and one can see from Figs. 3 and 4, that in this region the approximation of eq.(6) can give an appropriate expression for at least the electric form factor.

Some higher order structure dependent corrections were calculated by us in Ref. [65], and all old results were also reviewed there. That have allowed to give some prediction. According to a classical review by G. T. Bodwin and D. R. Yennie, 1988 [66], the comparison of theory and experiment leads for the hydrogen hyperfine splitting to

$$\frac{\nu_{hfs}(exp) - \nu_{hfs}(theo)}{\nu_{hfs}(exp)} = (0.48 \pm 0.56) \text{ ppm.}$$

Proton polarizability is not included in $\nu_{hfs}(theo)$ and the difference above has to be interpreted as its contribution. The theoretical limitation for the proton polarizability contribution is [67]

$$\left| \frac{\nu_{hfs}(exp) - \nu_{hfs}(theo)}{\nu_{hfs}(exp)} \right| < 4 \text{ ppm.}$$

That is obtained from a treatment of the inelastic scattering data. Our result for the comparison was [65]

$$\frac{\nu_{hfs}(exp) - \nu_{hfs}(theo)}{\nu_{hfs}(exp)} = (3.4 \pm 0.9) ppm.$$

Due to the status of the magnetic radius which is not good determined we need to say here that the uncertainty can be larger by factor like 3 from the pure extrapolation data in the experimental work [2]. Using the dispersion fitting data [6] the result has to be shifted slightly, but the uncertainty has to be approximately the same (0.9 ppm), because $R_E = 0.847(9)$ fm and $R_M = 0.853(9)$ fm and so the radii are equal one to the other within their uncertainty. According the Wong fit the result for the polarizability is rather close to 4(2) ppm within assumption, that the magnetic radius is equal to the electric one.

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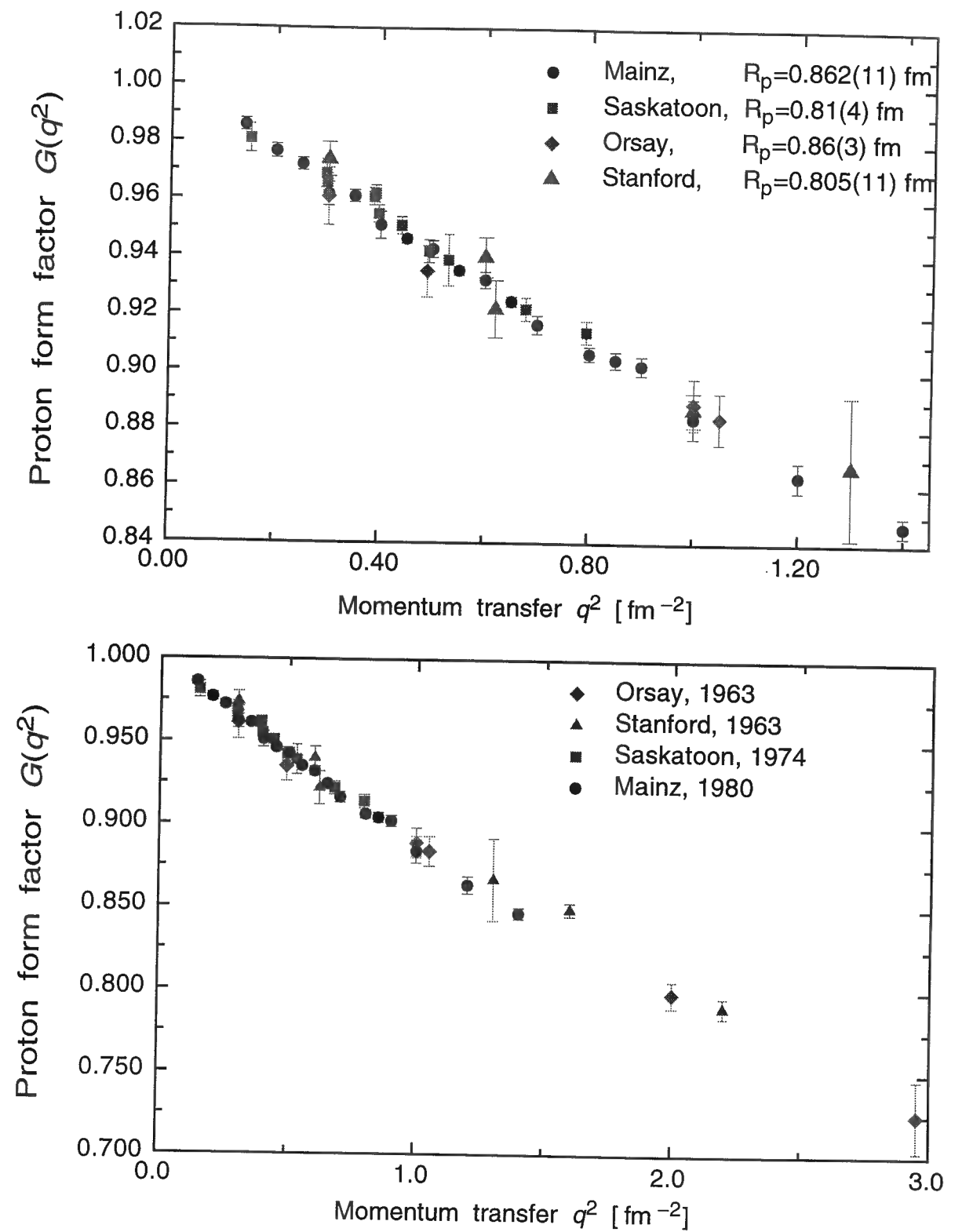


Figure 3: Experimental data of the elastic electron-proton scattering are given for the whole low energy range (a), and they are only for the Mainz-1980 fitting momentum range (b)

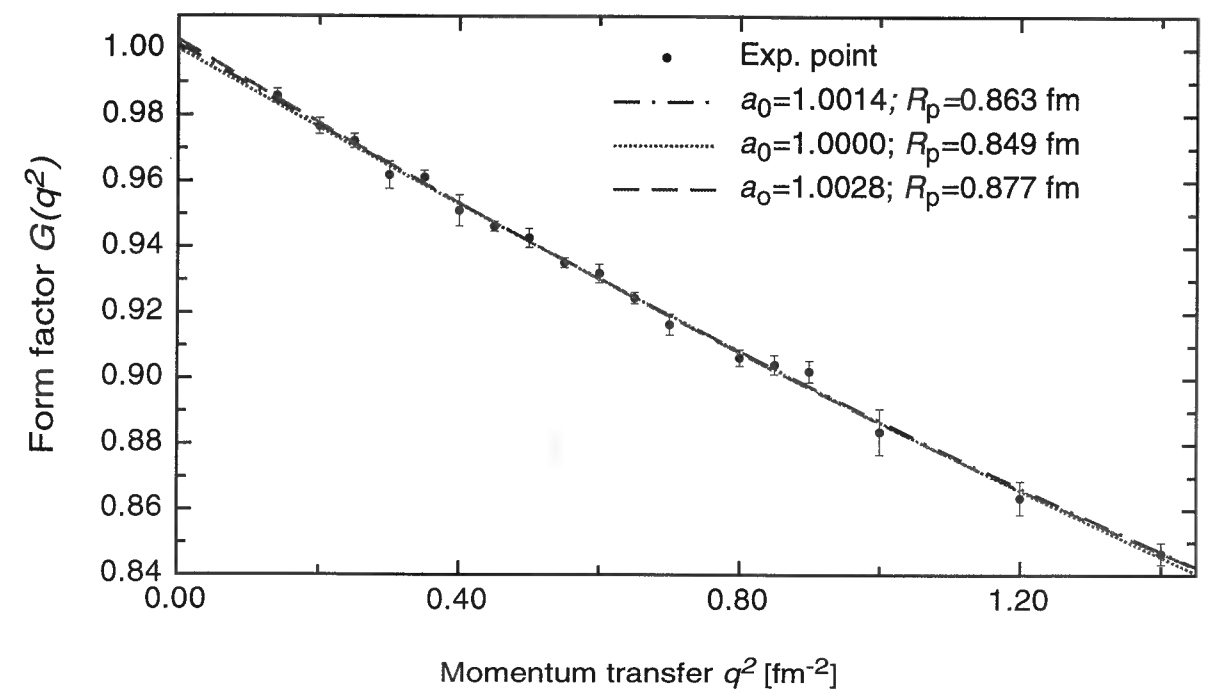
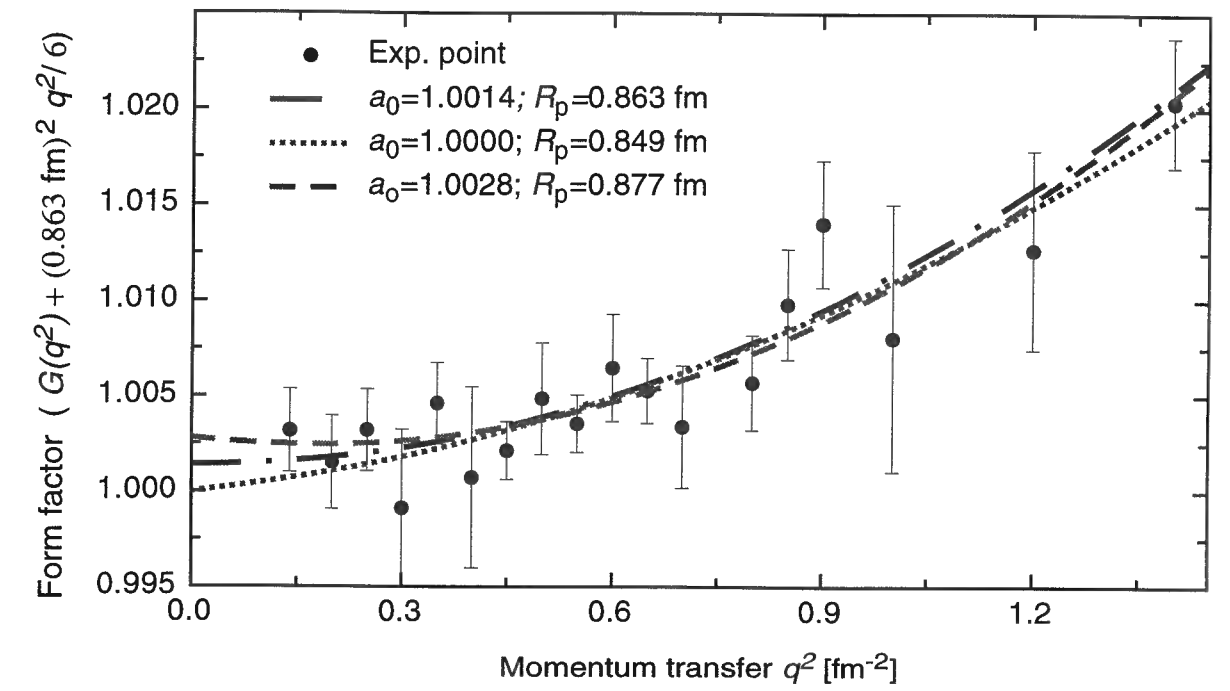


Figure 4: The Wong fits [58] of the Mainz-1980 data [2]

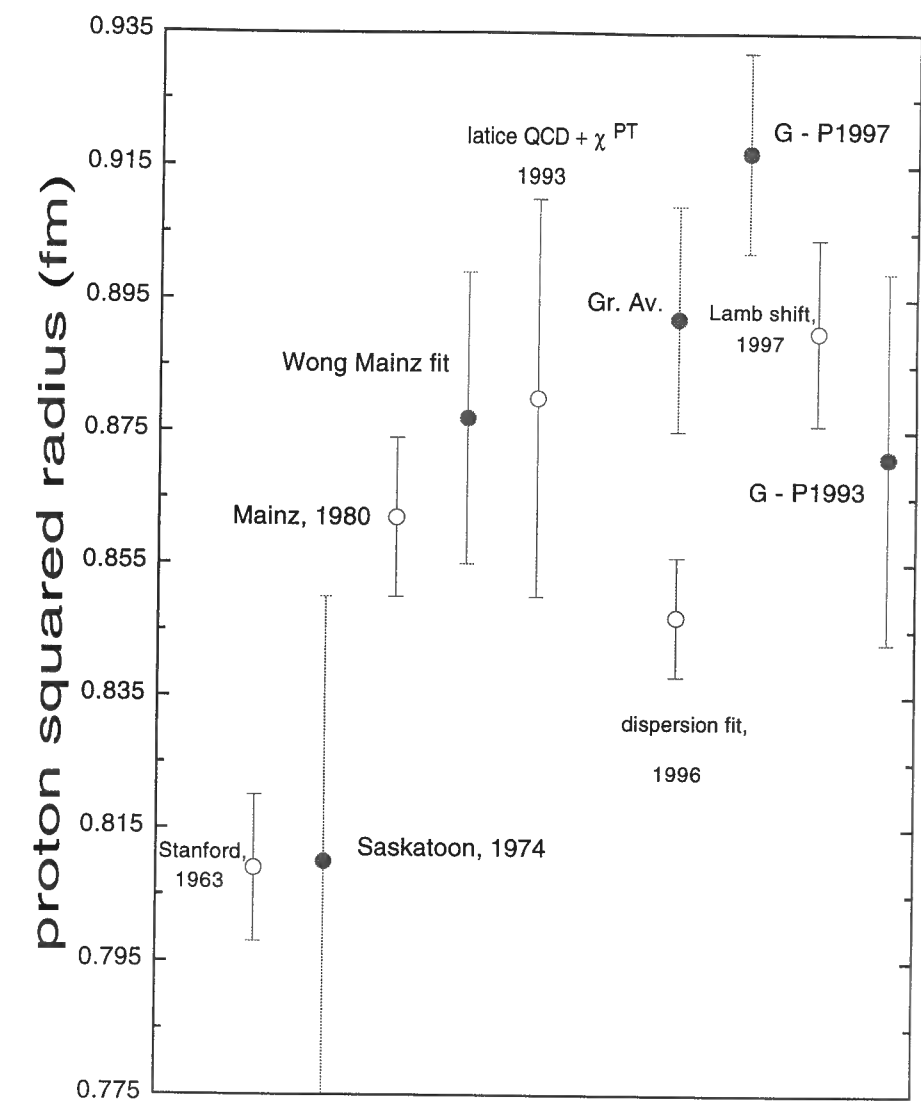


Figure 5: The different values of the proton radius as it has to be

