# III.5 Mixed state entanglement



## a) Introduction

When is a mixed state $\rho_{AB}$ entangled?

i) If $\rho_{AB}$ cannot be created by LOCC (or $\rho_{AB}^{\otimes N}$)

ii) If we can extract ebits from $\rho_{AB}$ or $\rho_{AB}^{\otimes N}$

(iii) If it helps us to perform some task better in an LOCC setting.

Clearly, there are more states w/ (i) than (iii) than (ii)
"$\geq$" "$\geq$"

Use (i) as our definition:

States which <u>can</u> be prepared by LOCC:

$$(*)\quad \boxed{\rho = \sum p_i\, \rho_i^A \otimes \rho_i^B \qquad \text{"separable state"}}$$

$$\left(\rho_i^A, \rho_i^B \geq 0,\ p_i \geq 0;\ \sum p_i = 1,\ \mathrm{tr}\,\rho_i^A = \mathrm{tr}\,\rho_i^B = 1\right)$$

We define: $\boxed{\rho \text{ is entangled if it is not separable}}$

(i.e., it <u>cannot</u> be written in the form $(*)$).

Given $\rho$, how can we check if it is sep./entangled?

<u>Problem:</u> Need to check over all decompositions

$$\rho \overset{?}{=} \sum p_i \rho_i^A \otimes \rho_i^B$$

Ambiguity in ensemble decomposition = isometries: need to optimize over isometries!

(Note: can always choose $\rho_i^\bullet$ pure by further decomp. $\rho_i^\bullet$).

In fact, general problem is NP-hard ("exponentially" hard in dimension of space)

## 6) Entanglement witnesses

Structure of set of separable states:

Let $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$ ; $\sigma = \sum q_j \sigma_j^A \otimes \sigma_j^B$

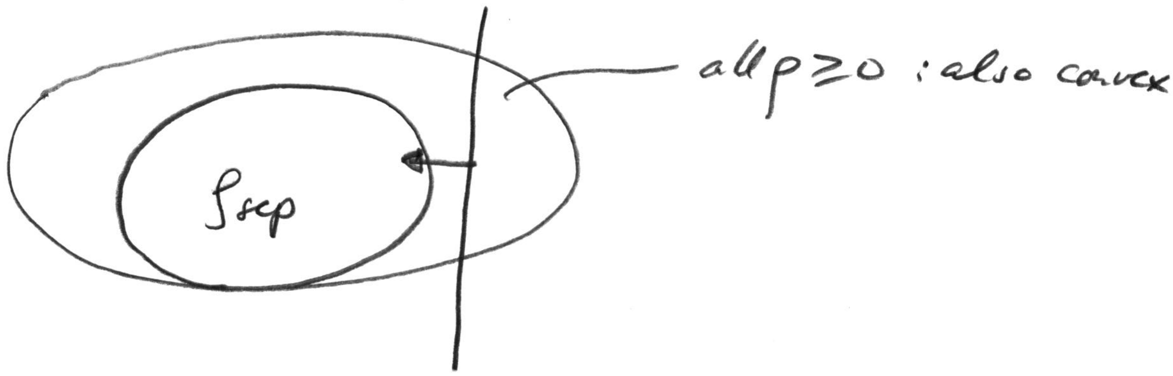
$$\Rightarrow \lambda\rho + (1-\lambda)\sigma = \sum r_k \chi_k^A \otimes \chi_k^B \ , \ \lambda \in [0;1]$$

with $r_k = (\lambda p_1, \lambda p_2, \dots, (1-\lambda) q_1, (1-\lambda) q_2, \dots)$

and $\chi_k^{A/B} = (\rho_1^{A/B}, \rho_2^{A/B}, \dots, \sigma_1^{A/B}, \sigma_2^{A/B}, \dots)$

$\Rightarrow \lambda\rho + (1-\lambda)\sigma$ separable

$\Rightarrow$ separable states form <u>convex set</u>!




Can find <u>hyperplanes</u> s.th. all $\rho$ on one side are entangled!

General hyperplane is of form $tr[X\rho] + c = 0$.

$$\Leftrightarrow tr[\underbrace{(X + c\,\mathbb{1})}_{=:W}\rho] = 0.$$

$=: W \rightarrow$ can be hermitian!

two sides of hyperplane $\Leftrightarrow tr[W\rho] \gtrless 0$.

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Entanglement witness: $W = W^\dagger$ s.th.
$tr[W\rho] \geq 0 \;\forall \rho$ separable, i.e.
$tr[W\rho] < 0 \Rightarrow \rho$ entangled!

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Notes:

- Need ways to show that $\rho$ sep. $\Rightarrow \mathrm{tr}[W\rho] \geq 0$ !
- Witness can only detect <u>certain ent. states</u>!
- Convex set char. by all tangent planes:

  $\Rightarrow \exists$ witness for <u>any</u> ent. state.
- Witness <u>linear operator</u> $\Rightarrow$ experimentally measurable!

Example:

$$W = \mathbb{F} \quad (\text{"flip"}); \qquad \mathbb{F} := \sum_{i,j=1}^{d} |i,j\rangle\langle j,i|$$

"magic formula": $\underline{\mathrm{tr}[\mathbb{F}(A\otimes B)]} = \sum_{i,j} \mathrm{tr}[\,|i,j\rangle\langle j,i|\, A\otimes B\,]$

$$= \sum \langle j,i|\, A\otimes B\,|i,j\rangle = \sum A_{ji} B_{ij}$$

$$= \underline{\mathrm{tr}(AB)}$$

Let $\rho_{sep} = \sum p_i\, \rho_i^A \otimes \rho_i^B$:

$$\mathrm{tr}[W\rho_{sep}] = \sum p_i\, \mathrm{tr}\left(\mathbb{F}(\rho_i^A \otimes \rho_i^B)\right)$$

$$= \sum p_i\, \underbrace{\mathrm{tr}[\rho_i^A \rho_i^B]}_{\geq 0} \geq 0$$

$\geq 0$: $P, Q \geq 0$, $P = \sum \lambda_i |\varphi_i\rangle\langle\varphi_i|$:

$\Rightarrow \mathrm{tr}[PQ] = \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{\langle\varphi_i|Q|\varphi_i\rangle}_{\geq 0} \geq 0.$

$\Rightarrow$ W is ent. witness.

Which states does W detect? $\rightarrow$ W detect anti-sym. states!

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \Rightarrow \mathbb{F}|\psi^-\rangle = -|\psi^-\rangle \Rightarrow \langle\psi^-|\mathbb{F}|\psi^-\rangle = -1,$$

while for sym. ME state, $\mathbb{F}|\chi\rangle = |\chi\rangle \Rightarrow \langle\chi|\mathbb{F}|\chi\rangle = +1$ !
(i.e.: $|\chi\rangle = |\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$)

<u>Mixed states:</u>

$$\rho = \lambda|\psi^-\rangle\langle\psi^-| + (1-\lambda)\frac{\mathbb{1}}{4}, \quad \lambda \in \left[-\tfrac{1}{3}, 1\right] : \text{"Werner state"}$$

$$\mathrm{tr}[\mathbb{F}\rho] = \lambda\underbrace{\langle\psi^-|\mathbb{F}|\psi^-\rangle}_{=-1} + (1-\lambda)\underbrace{\mathrm{tr}\left(\frac{\mathbb{1}}{4}\mathbb{F}\right)}_{=1/2} = \frac{1}{2}(1-3\lambda)$$

$\Rightarrow$ State entangled if $\lambda \geq \frac{1}{3}$.

<u>Optimal?</u> Yes, e.g. $\rho = |0\rangle\langle 0| \otimes |1\rangle\langle 1| \Rightarrow \mathrm{tr}[\mathbb{F}\rho] = 0$
(= touches convex set!)

Other witnesses: e.g. $W = \mathbb{1} - d|\Omega\rangle\langle\Omega|$; $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i,i\rangle$

$\longrightarrow$ Homework!

## c) Positive maps and the PPT criterion

Reminder: A superoperator $\Lambda$ is called positive if

$$\rho \geq 0 \Rightarrow \Lambda(\rho) \geq 0.$$

Usually we require $\Lambda$ to be completely positive, i.e. $(\Lambda \otimes \mathbb{1})(\rho) \geq 0$ for $\rho \geq 0$. Here, we will however be interested in positive but not compl. positive maps!

Why? - Consider $\rho_{sep} = \sum_i p_i \rho_i^A \otimes \rho_i^B$.

$$(\Lambda \otimes \mathbb{1})(\rho_{sep}) = \sum_i p_i \underbrace{\Lambda(\rho_i^A)}_{=\tilde{\rho}_i^A \geq 0} \otimes \mathbb{1}(\rho_i^B) = \sum p_i \tilde{\rho}_i^A \otimes \rho_i^B \geq 0$$

i.e.: $$(\Lambda \otimes \mathbb{1})(\rho) \ngeq 0 \Rightarrow \rho \text{ entangled}$$

"has negative eigenvalues"

Most important example:

$$\Lambda(\rho) := \rho^T \quad (\text{transpose})$$

$$(\Lambda \otimes \mathbb{1})(\rho) \equiv \rho^{T_A} \quad : \text{"partial transpose"}$$

(i.e.: $\rho = \sum \rho_{ij}^{i'j'} |i,j\rangle\langle i',j'| \Rightarrow \rho^{T_A} = \sum \rho_{ij}^{i'j'} |i',j\rangle\langle i,j'|$ )

We have thus:

$$\boxed{\begin{array}{l} \rho^{T_A} \not\geq 0 \Rightarrow \rho \text{ entangled} \\ \text{"PPT (positive partial transpose) criterion"} \end{array}}$$

E.g.: $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$

$$\Rightarrow \left(|\Omega\rangle\langle\Omega|\right)^{T_A} = \frac{1}{d} \sum \left(|i,i\rangle\langle j,j|\right)^{T_A} = \frac{1}{d} \underbrace{\sum |j,i\rangle\langle i,j|}_{\equiv \mathbb{F}}$$

Not positive: antisym. states have neg. eigenvalue, e.g.

$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$:

$$\langle\phi^-|\left(|\Omega\rangle\langle\Omega|\right)^{T_A}|\phi^-\rangle = \frac{1}{2d}\left(\langle 01| - \langle 10|\right)\left(|10\rangle - |01\rangle\right) = \frac{-1}{2d}$$

Mixed state example:

$$\rho = \lambda |\Omega\rangle\langle\Omega| + (1-\lambda)\frac{\mathbb{1}}{d^2}, \quad \frac{-1}{d^2-1} \leq \lambda \leq 1$$

"isotropic state"

E.g. for $d=2$:
$$\rho = \lambda \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & & & 0 \\ 0 & & & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + (1-\lambda)\begin{pmatrix} 1/4 & & & \\ & 1/4 & & \\ & & 1/4 & \\ & & & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\lambda}{4} & & & \frac{\lambda}{2} \\ & \frac{1-\lambda}{4} & 0 & \\ & 0 & \frac{1-\lambda}{4} & \\ \frac{\lambda}{2} & & & \frac{1+\lambda}{4} \end{pmatrix}$$

$$\Rightarrow \rho^{T_A} = \begin{pmatrix} \frac{1+\lambda}{4} & & & \\ & \frac{1-\lambda}{4} & \frac{\lambda}{2} & \\ & \frac{\lambda}{2} & \frac{1-\lambda}{4} & \\ & & & \frac{1+\lambda}{4} \end{pmatrix}$$

$\Rightarrow$ positive if & only if $\frac{\lambda}{2} \leq \frac{1-\lambda}{4} \Rightarrow \lambda \leq \frac{1}{3}$.

Note: Criterion indep. of local unitaries on B $\Rightarrow$ detects all max. ent. states! ($\to$ stronger than witnesses!)

In fact: PPT criterion detects all entangled states in dimension $d_A \times d_B = 2\times 2$ and $3\times 2$
(but counterex. exist in $3\times 3$, $4\times 2$: "PPT bound ent. states")

Other example: $\Lambda(\rho) = tr(\rho)\mathbb{1} - \rho$:

$$(\Lambda\otimes\mathbb{1})(\rho) = \underbrace{tr\rho}_{=1}\cdot(\mathbb{1}\otimes tr_A\rho) - \rho = \mathbb{1}\otimes\rho_B - \rho \not\geq 0$$

$\Rightarrow \rho$ entangled.

"reduction criterion" $\mathbb{1}\otimes tr_A\rho \not\geq \rho$ ($\to$ homework)

## d) Relation of witnesses & positive maps



For each witness $W$, there is a positive map $\Lambda$ which detects all states $W$ detects (and in fact more).

interpret witness $W$ as "bipartite state" (better: operator)

map $\Lambda \hat{=}$ Jamiołkowski map of "state" $W^T$:

$$\Lambda(X) := \mathrm{tr}_A \left( W^T (X^T \otimes \mathbb{1}) \right) = \mathrm{tr}_A \left( W (X \otimes \mathbb{1}) \right)^T .$$

(Note: $A \leftrightarrow B$ swapped w.r.t. Chapter II)

Then, $\langle \phi | \Lambda(\rho)^T | \phi \rangle = \langle \phi | \mathrm{tr}_A (W(\rho \otimes \mathbb{1})) | \phi \rangle$

$$= \mathrm{tr}_A \Big( W \big( \underbrace{\rho \otimes |\phi\rangle\langle\phi|}_{\text{sep.}} \big) \Big) \geq 0 \quad \text{for } \rho \geq 0,$$

i.e., $\Lambda$ is a <u>positive map</u>, and

$$\mathrm{tr}\left[ W(A \otimes B) \right] = \mathrm{tr}_B \Big[ \mathrm{tr}_A \big( W (A \otimes \mathbb{1}) \big) B \Big] = \mathrm{tr}\left[ \Lambda(A)^T B \right]$$

$$= \sum_{ij} \left[ \Lambda(A)^T \right]_{ij} B_{ji} = d \langle \Omega | \Lambda(A) \otimes B | \Omega \rangle$$

$$= d \langle \Omega | (\Lambda \otimes \mathbb{1})(A \otimes B) | \Omega \rangle$$

Linearity: $\mathrm{tr}(W\rho) = d \langle \Omega | (\Lambda \otimes \mathbb{1})(\rho) | \Omega \rangle$

i.e.: $\mathrm{tr}(W\rho) < 0 \Rightarrow (\Lambda \otimes \mathbb{1})(\rho) \not\geq 0$

$\Rightarrow$ $\Lambda$ detects all states which $W$ detects!

e.g.: $W = \mathbb{F}$:

$$\Lambda(X) = \mathrm{tr}_A\left(\mathbb{F}(X^T \otimes \mathbb{1})\right) = \mathbb{1} \cdot X^T \cdot = X^T$$

$\Rightarrow$ PPT criterion!

Note: PPT strictly stronger: $\mathbb{F}$ could e.g. not detect $|\Omega\rangle$!

Corollary: A state is separable if and only if

$(\Lambda \otimes \mathbb{1})(\rho) \geq 0$ for all positive $\Lambda$.

## e) Quantification of mixed state entanglement

How to quantify mixed state entanglement?

i) Entanglement needed to create state, e.g.

min. avg. amount of $E(|\phi\rangle) = S(\mathrm{tr}_B |\phi\rangle\langle\phi|)$ needed

"entanglement of formation"

$$E_F(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i E(|\psi_i\rangle)$$
$$\text{s.th. } \sum p_i |\psi_i\rangle\langle\psi_i| = \rho$$

"entanglement cost" $E_C(\rho) = \lim_{N\to\infty} \frac{1}{N} E_F(\rho^{\otimes N})$

(asympt. cost per copy)

ii) Extractable entanglement:

"Distillable entanglement" $E_D(\rho)$:

$E_D(\rho)$ = max. rate $R = \frac{M}{N}$ achievable w/ LOCC

protocol $\mathcal{E}_N$ s.th. $\mathcal{E}_N(\rho^{\otimes N}) \to |\Omega\rangle\langle\Omega|^{\otimes M}$

<u>Note:</u> $E_F \geq E_C \geq E_D$.

Generally $E_C(\rho) \gneq E_D(\rho)$: process not reversible,

no unique measure.

(E.g.: $\rho$ PPT, $\rho^{T_A} \geq 0$. LOCC map preserves PPT

$\Rightarrow$ PPT states are undistillable, $E_D(\rho) = 0$.

But for ent. PPT states $E_C(\rho) > 0$.

Converse question: $\rho^{T_A} \ngeq 0 \Rightarrow E_D(\rho) > 0$ is big

open problem - existence of "NPT bound entanglement")

Problem: $E_F$, $E_C$, $E_D$ very hard to compute!

→ want computable ent. measures.

Wishlist for ent. measures:

⇒ • LOCC-monotone: cannot be increased by LOCC ⇐

• 0 on sep. states

• additive: $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$

• continuous: $\rho \approx \sigma \Rightarrow E(\rho) \approx E(\sigma)$ (w/some $\delta$-$\varepsilon$)

• $E_D \leq E \leq E_C$.

• coincides with $E(|\psi\rangle) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$ on pure states.

(Almost) impossible to get all: LOCC monotonicity most important.

Negativity – a computable ent. measure

Have seen: $\rho^{T_A}$ has neg. eigenvalues ⇒ $\rho$ entangled.

Use neg. eigenvalues as ent. measure:

Negativity $\mathcal{N}(\rho) = \frac{1}{2}\Big(\underbrace{\sum_i |\lambda_i(\rho^{T_A})|}_{=:\|\rho^{T_A}\|_1} - 1\Big)$

$$= \frac{1}{2}(\|\rho^{T_A}\| - 1) = -\sum_{\text{neg. eigenvalues}} \lambda_i(\rho^{T_A}). \quad \left(\text{since } \mathrm{tr}(\rho^{T_A}) = 1\right)$$

or log-negativity $E_N(\rho) = \log_2 \| \rho^{T_A} \|_1$.

Properties:

Negativity $\mathcal{N}$:

- LOCC monotone
- 0 on sep. states, but can be 0 on ent. states
- $\neq E(|\psi\rangle)$ for pure states.
- not additive
- continuous

Log-negativity $E_N$:

- additive
- 0 on sep., but can be 0 on ent. states
- $\neq E(|\psi\rangle)$ for pure states.
- not an LOCC monotone (!)
- continuous