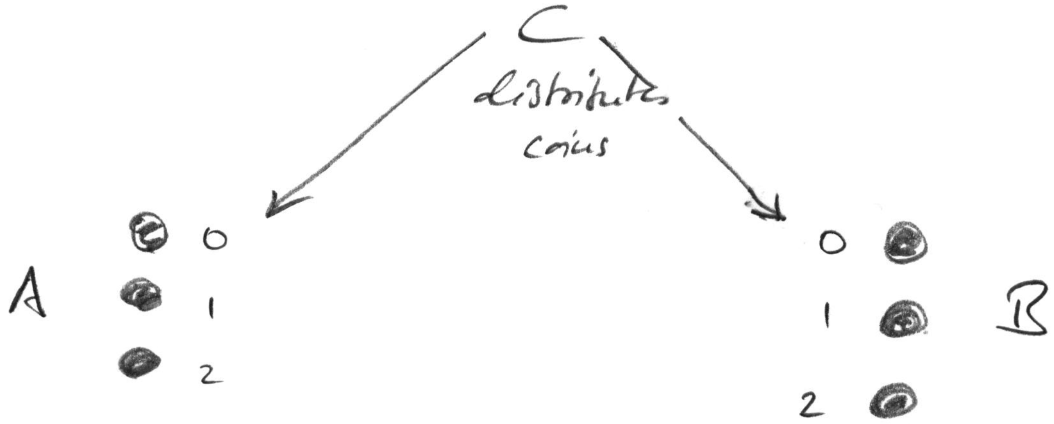
2. Bell inequalities

How non-classical are entangled states?

Consider the following game of A+B with coins:



- A+B each get 3 coins in boxes (labelled 0, 1, 2), prepared in some (deterministic or random) way by C.
- A & B can look at one coin each ($i=0,1,2$ & $j=0,1,2$). Result is heads $=+1$ or tails $=-1$. We denote result by $a_i = \pm 1$ and $b_j = \pm 1$.
- A & B observe: If they look at the same coin, they always get the same outcome: $\boxed{a_i = b_i}$

- Can A infer the value of 2 of her coins?

<u>Idea</u>: A looks at i, B at $j = i' \neq i$.

Since $a_{i'} = b_{i'}$, they now know a_i <u>and</u> $a_{i'}$.

Clearly works classically!

- What does this imply?

  - A & B can use this to estimate prob. $p(a_i = a_{i'})\ \forall i, i'$.

  - Clearly, we must have

$$p(a_0 = a_1) + p(a_1 = a_2) + p(a_2 = a_0) \geq 1,$$

since in each instance of the game, at least 2 coins must be equal.

$$\xRightarrow{a_{i'} = b_{i'}} \underline{\underline{p(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) \geq 1}} \quad (*)$$

is satisfied classically!

$(*)$ is called a <u>Bell inequality</u>.

<u>But</u>: In a quantum mechanical version of the game, the Bell inequality $(*)$ can be violated!

Q.M. version of game:

* C distributes an entangled state $|\psi\rangle$.

* A & B perform meas. which depends on $i/j \to a_i/b_j$.

Choose $|\psi\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

A & B will measure spin along some axes $\vec{u}_i$ and $\vec{w}_j$, i.e. the operators $\vec{u}_i \cdot \vec{\sigma}^A$ and $\vec{w}_j \cdot \vec{\sigma}^B$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

We have $(\vec{\sigma}^A + \vec{\sigma}^B)|\psi^-\rangle = 0$ (i.e. $(\sigma_\ell^A + \sigma_\ell^B)|\psi^-\rangle = 0 \ \forall \ell$)

Then, $\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\underbrace{\vec{\sigma}^B}_{\vec{\sigma}^B|\psi^-\rangle = -\vec{\sigma}^A|\psi^-\rangle} \cdot \vec{w})|\psi^-\rangle =$

$$= -\langle\psi^-|(\vec{\sigma}^A \cdot \vec{u})(\vec{\sigma}^A \cdot \vec{w})|\psi^-\rangle$$

$$= \sum_{k\ell} u_k w_\ell \underbrace{\mathrm{tr}(\underbrace{\rho_A}_{=\frac{1}{2}\mathbb{1}} \sigma_k^A \sigma_\ell^A)}_{=\delta_{k\ell}} = -\sum_k u_k w_k = -\vec{u}\cdot\vec{w}$$

$$= -\cos\theta$$

angle betw. $\vec{u}$ & $\vec{w}$.

Measurement of A/B along $\vec{n}/\vec{m}$:

$\longrightarrow$ projections $E_{\pm 1}(\vec{u}) = \frac{1}{2}(\mathbb{1} \pm \vec{u}\cdot\vec{\sigma})$

$$p(\pm 1, \pm 1) = \langle \psi^- | E_{\pm 1}(\vec{u})\, E_{\pm 1}(\vec{m}) | \psi^- \rangle$$

$$= \frac{1}{4} \langle \psi^- | \underbrace{\mathbb{1}}_{\hat{=}1} \pm \underbrace{\vec{u}\cdot\vec{\sigma}^A}_{\hat{=}0} \pm \underbrace{\vec{m}\cdot\vec{\sigma}^B}_{\hat{=}0} + \underbrace{(\vec{u}\cdot\vec{\sigma}^A)(\vec{m}\cdot\vec{\sigma}^B)|\psi^-\rangle}_{\hat{=}-\cos\Theta}$$

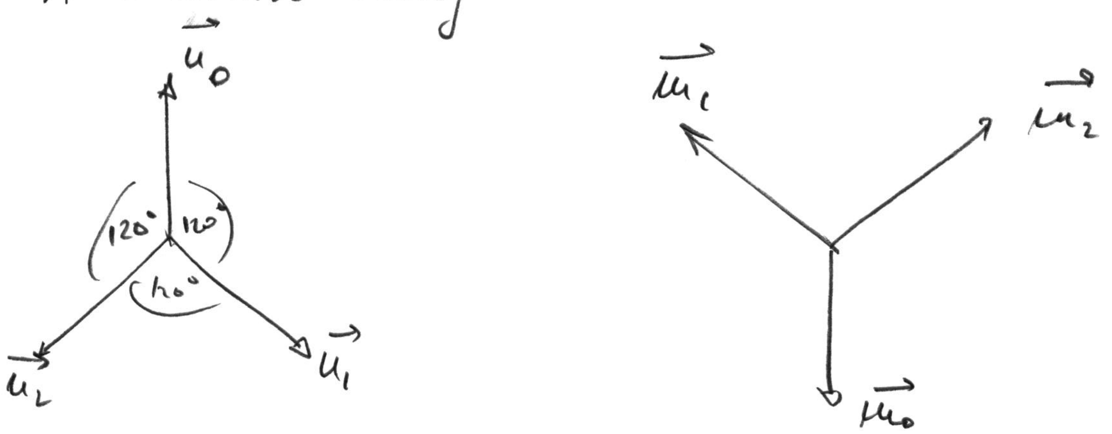
$$= \frac{1}{4}(1 - \cos\Theta)$$

and

$$p(\pm 1, \mp 1) = \frac{1}{4}(1 + \cos\Theta).$$

$\Longrightarrow$ $P_{equal} = \frac{1}{2}(1 - \cos\Theta)$, $P_{different} = \frac{1}{2}(1 + \cos\Theta)$.

Now let A measure along



in the XZ-plane, and B along $\vec{m}_i = -\vec{u}_i$.

(48)

$\boxed{\text{(49) + (50) missing}}$

- $i = j$: $P_{equal} = \frac{1}{2}(1 - \cos 180^\circ) = 1$ ✓
  (same basis for A & B)

- $i \neq j$: $P_{equal} = \frac{1}{2}\Big(1 - \underbrace{\cos(\pm 60^\circ)}_{=\frac{1}{2}}\Big) = \frac{1}{4}$
  (diff. basis for A & B)

$\Rightarrow P(a_0 = b_1) + p(a_1 = b_2) + p(a_2 = b_0) = \frac{3}{4} < 1$

$\Rightarrow$ Bell inequality violated.

Assumptions:

① Realism: Outcomes of measurements are "elements of reality" (i.e., have pre-determined values) even without measurement.

② Locality: A & B's boxes cannot communicate once distributed.

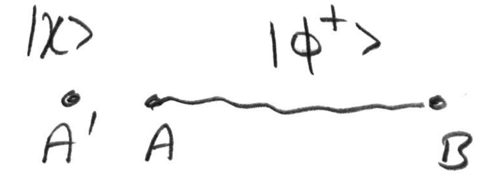
$\Rightarrow$ QM predictions incompatible with local and realistic description.

$\Rightarrow$ We need to give up either locality or realism (or both).

3. Applications of entanglement: Teleportation, dense coding

Teleportation:

Setup:



A&B share ent. state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$

A has unknown state $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$

(Note: could be part of larger system (→ linearity!))

A&B cannot send q. states, but can communicate classically "for free"

<u>Can A get $|\chi\rangle$ to B?</u>

Measurement of $|\chi\rangle$ would <u>break</u> state (destroy info ↯

⇒ <u>Teleportation!</u>

(<u>Motivation</u>: Transmitting q. info is subject to <u>noise</u> → info could be destroyed. w/ telep., we can first build up $|\phi^+\rangle$ – by retrying or ent. distillation (→ later) & then teleport: <u>noise-free</u> transmission line!)

Protocol:

1. A performs meas. in <u>Bell basis</u>

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = (Z\otimes I)|\phi^+\rangle = (I\otimes Z)|\phi^+\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = (X\otimes I)|\phi^+\rangle = (I\otimes X)|\phi^+\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = (ZX\otimes I)|\phi^+\rangle = (I\otimes XZ)|\phi^+\rangle$$

We also write $|\phi_{\alpha\beta}\rangle = (Z^\alpha X^\beta \otimes I)|\phi^+\rangle = (I\otimes X^\beta Z^\alpha)|\phi^+\rangle$

$(\alpha,\beta = 0,1)$.

Outcome probabilities for $|\phi_{\alpha\beta}\rangle$:

$$\rho_A = \mathrm{tr}_B\left[|\phi^+\rangle\langle\phi^+|_{AB}\right] = \frac{1}{2}\mathbb{1}.$$

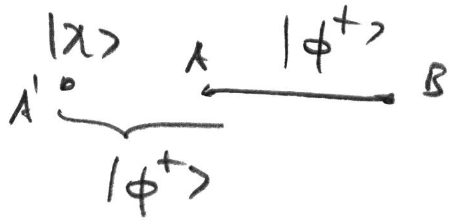
$$\langle\phi_{\alpha\beta}|\,|\chi\rangle\langle\chi|_{A'}\otimes\frac{1}{2}\mathbb{1}_A\,|\phi_{\alpha\beta}\rangle = \frac{1}{2}\mathrm{tr}\left[\left(|\chi\rangle\langle\chi|_{A'}\otimes\mathbb{1}\right)|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]$$

$$= \frac{1}{2}\mathrm{tr}_{A'}\Big[|\chi\rangle\langle\chi|_{A'}\cdot\underbrace{\mathrm{tr}_A\left[|\phi_{\alpha\beta}\rangle\langle\phi_{\alpha\beta}|\right]}_{=\frac{1}{2}\mathbb{1}}\Big] = \frac{1}{4}.$$

$\Rightarrow$ equal prob. for all 4 outcomes, <u>indep. of $|\chi\rangle$</u>.
(Good: No info about $|\chi\rangle$ acquired $\Rightarrow$ no disturbance.)

What is state of B after meas.?

i) Outcome $|\phi^+\rangle = |\phi_{00}\rangle$:



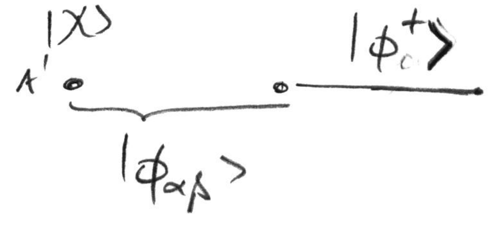
$$\langle\phi^+|_{A'A}\left(|\chi\rangle_{A'}\otimes|\phi^+\rangle_{AB}\right) =$$

$$= \frac{1}{2}\left(\langle 00|_{A'A}+\langle 11|_{A'A}\right)\left(\left(\alpha|0\rangle_{A'}+\beta|1\rangle_{A'}\right)\left(|00\rangle_{AB}+|11\rangle_{AB}\right)\right)$$

$$= \underline{\underline{\frac{1}{2}\left(\alpha|0\rangle_B+\beta|1\rangle_B\right)}}.$$

<u>State $|\chi\rangle$ appears at B!</u>

ii) General outcome:



$$\langle\phi_{\alpha\beta}|_{A'A}\ |\phi^+\rangle_{AB} = \langle\phi^+|_{A'A}\left(\mathbb{1}_{A'}\otimes Z_A^\alpha X_A^\beta\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\cdot\left(Z_A^\alpha X_A^\beta\otimes\mathbb{1}_B\right)|\phi^+\rangle_{AB}$$

$$= \langle\phi^+|_{A'A}\cdot\left(\mathbb{1}_A\otimes X_B^\beta Z_B^\alpha\right)|\phi^+\rangle_{AB}$$

$$\Rightarrow \left(\langle\phi_{\alpha\beta}|_{A'A}\right)\left(|\chi\rangle_{A'} \otimes |\phi^+\rangle_{AB}\right)$$

$$= X_B^{\beta} Z_B^{\alpha} \cdot \underbrace{\left(\langle\phi_{\alpha\beta}|_{A'A}\, |\chi\rangle_{A'} \otimes |\phi^+\rangle_{AB}\right)}_{=|\chi\rangle_B.}$$

$$= X^{\beta} Z^{\alpha} |\chi\rangle$$

$\Rightarrow$ outcome is $X^{\beta} Z^{\alpha} |\chi\rangle$ w/ prob. $\frac{1}{4}$ each.

$\Rightarrow$ avg. state of B is $\frac{1}{4}\sum X^{\beta} Z^{\alpha} |\chi\rangle\langle\chi| Z^{\alpha} X^{\beta} = \frac{1}{2}\mathbb{1}$.

$\Rightarrow$ no information at Bob's side!

(2) A communicates meas. outcome (α,β) to Bob, & B applies $(X^{\beta}Z^{\alpha})^{+}$

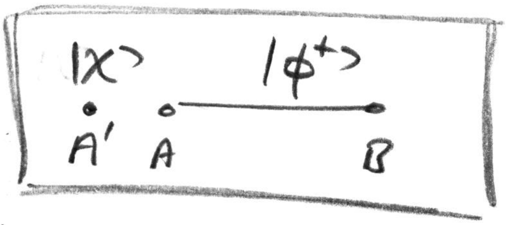
$\Rightarrow$ Bob recovers $|\chi\rangle$.

Notes: * No faster-than-light communication!

* Communicating 1 qubit requires 1 "e-bit" (max. ent. state of 1+1 qubit) + 2 bits of class. communication

Teleportation protocol:

① Measure A, A' in $|\phi_{\alpha\beta}\rangle$ basis.



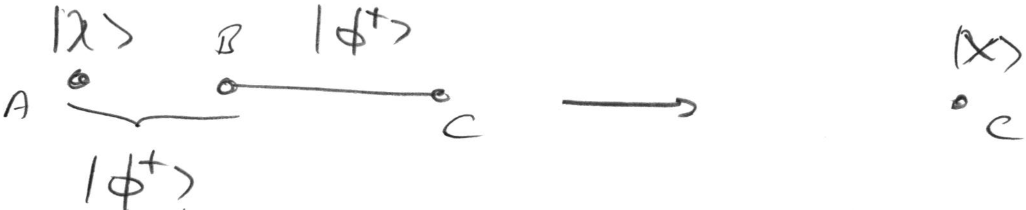
② Communicate (α, β) from A to B.

③ Perform $(X^\beta Z^\alpha)^\dagger$ on B.

Generalization to qu-d-its straightforward! ($\rightarrow$ HW)

Relation betw. teleportation & Choi-Jamiolkowski:

① Consider "postselected teleportation"

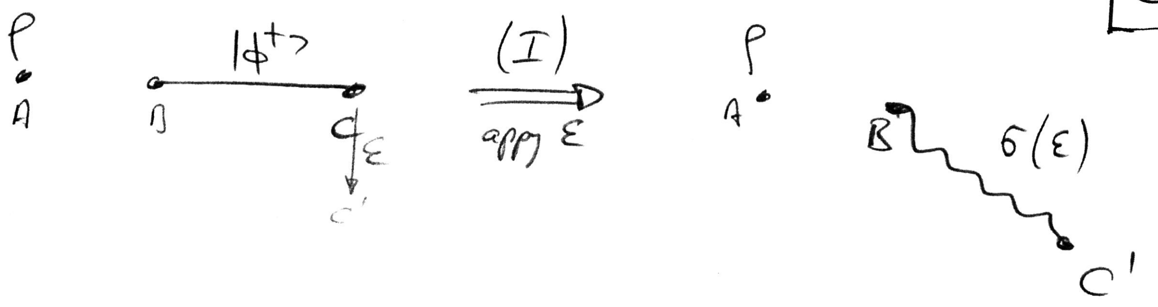


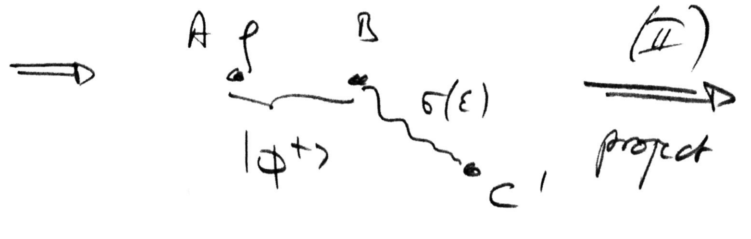
projection onto $|\phi^+\rangle$ $\equiv$ "postselected" meas.

② Protocol for applying $\rho \mapsto \mathcal{E}(\rho)$:

57 missing

3 Interchg. order of proj. & appl. of $\mathcal{E}$:





$\tau = \mathcal{E}(\rho)$, since order commutes!

This is Choi-Jamiołkowski:

(I) is the $\mathcal{E} \longmapsto \sigma$ map, and

(II) is the $\sigma \longmapsto \mathcal{E}$ map.

(check formulas → HW)