

Kraus representation:

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We call

$$\mathcal{E}(\rho) = \sum \Pi_n \rho \Pi_n^\dagger; \quad \sum \Pi_n^\dagger \Pi_n = \mathbb{1}$$

the Kraus form or Kraus representation of \mathcal{E} .

Note: Any such \mathcal{E} can be realized by ancilla + unitary (cf. POVM). In fact, \mathcal{E} can be seen as POVM where we ignore result (meas. by environment).

Is this the most general physical evolution?

Conditions for physical evolution \mathcal{E} :

- (i) hermiticity-preserving: $\rho = \rho^\dagger \Rightarrow \mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger$
- (ii) positive: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) \geq 0$.
- (iii) trace-preserving: $\text{tr}(\rho) = 1 \Rightarrow \text{tr}(\mathcal{E}(\rho)) = 1$
- (iv) linear: $\mathcal{E}(\rho + \lambda \sigma) = \mathcal{E}(\rho) + \lambda \mathcal{E}(\sigma)$

(Note: w/out linearity, ensemble interpretation breaks down \rightarrow HW)

Is this sufficient? NO!

→ E should still be a physical map when it acts on part of a larger system.

(v) complete positivity:

$$\rho_{AB} \geq 0 \Rightarrow (E_A \otimes \mathbb{1}_B)(\rho_{AB}) \geq 0$$

(Note: $E_A \otimes \mathbb{1}_B$ def. on basis: $(E_A \otimes \mathbb{1}_B)(\pi \otimes N) = E_A(\pi) \otimes N$ + linearity)

We call E satisfying (i) - (v) a completely positive trace preserving (CPTP) map, or a quantum channel.

Are there maps which are positive (i) - (iv) but not CP?

YES: E.g. "transposition channel"

$$E(\rho) = \rho^T$$

$$(E \otimes \mathbb{1})(\rho_{AB}) = \rho_{AB}^{T_A} \text{ "partial transpose"}$$

Choi-Jamiołkowski isomorphism

Let $\mathcal{E} = \{ \mathcal{E} \mid \mathcal{E} \text{ CPTP} \}$ the space of CPTP maps on the density operators on \mathbb{C}^d , and

$$\mathcal{F} := \left\{ \sigma_{AB} \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d) \mid \sigma_{AB} \geq 0, \text{tr}_A(\sigma_{AB}) = \frac{1}{d} \mathbb{1} \right\}.$$

linear operators on $\mathbb{C}^d \otimes \mathbb{C}^d$

Then, the map

$$\hat{X}: \mathcal{E} \mapsto \sigma_{AB} = (\mathcal{E} \otimes \mathbb{1}_B)(|\Omega\rangle\langle\Omega|), \quad |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle|i\rangle$$

defines an isomorphism between \mathcal{E} and \mathcal{F} , the Choi-Jamiołkowski isomorphism, with σ_{AB} the Choi state of \mathcal{E} . The inverse map is given by

$$\hat{Y}: \sigma_{AB} \mapsto \mathcal{F}; \text{ with } \mathcal{F}(\rho) = d \cdot \text{tr}_B[\sigma_{AB} \cdot (\mathbb{1} \otimes \rho^T)]$$

Proof: We need to show

- (i) $\hat{Y}\hat{X} = \mathbb{1}_{\mathcal{E}}$
- (ii) $\hat{X}\hat{Y} = \mathbb{1}_{\mathcal{F}}$
- (iii) $\text{Im } \hat{X} = \{ \hat{X}(\mathcal{E}) \mid \mathcal{E} \in \mathcal{E} \} \subset \mathcal{F}$
- (iv) $\text{Im } \hat{Y} \subset \mathcal{E}$.

$$(i) \quad \hat{Y}(\hat{X}(\varepsilon)) / \rho = d \cdot \text{tr}_B \left[\underbrace{\hat{X}(\varepsilon)}_{\equiv \sigma_{AB}} \cdot (\mathbb{1}_A \otimes \rho^T) \right]$$

$$= \text{tr}_B \left[((\varepsilon \otimes \mathbb{1}_B) (|i\rangle\langle j|)) (\mathbb{1}_A \otimes \rho^T) \right]$$

$$= \sum_{ij} \text{tr}_B \left[((\varepsilon \otimes \mathbb{1}_B) (|i\rangle\langle j| \otimes |i\rangle\langle j|)) \cdot (\mathbb{1}_A \otimes \rho^T) \right]$$

$$= \sum_{ij} \varepsilon (|i\rangle\langle j|) \cdot \underbrace{\text{tr}_B [|i\rangle\langle j| \rho^T]}_{= \langle j | \rho^T | i \rangle = \rho_{ij}}$$

$$= \varepsilon \left(\sum_{ij} \rho_{ij} |i\rangle\langle j| \right) = \underline{\underline{\varepsilon(\rho)}} \quad \checkmark$$

$$(ii) \quad \hat{X}(\hat{Y}(\sigma_{AB})) = (\hat{Y}(\sigma_{AB}) \otimes \mathbb{1}) (|i\rangle\langle j|)$$

$$= \frac{1}{d} \sum_{ij} \underbrace{\hat{Y}(\sigma_{AB})}_{\equiv F} (|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \frac{1}{d} \sum_{ij} d \cdot \text{tr}_B \left[\sigma_{AB} \cdot (\mathbb{1} \otimes (|i\rangle\langle j|)^T) \right] \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \langle i | \sigma_{AB} | j \rangle_B \otimes |i\rangle_B \langle j|_B = \underline{\underline{\sigma_{AB}}} \quad \checkmark$$

(iii) $\sigma_{AB} = \hat{X}(\varepsilon) \geq 0$ by construction (ε CPTP).

$$\text{tr}_A(\sigma_{AB}) = \frac{1}{d} \sum_{i,j} \underbrace{h_A[\varepsilon(|i\rangle\langle j|) \otimes |i\rangle\langle j|]}_{\text{tr } \varepsilon(|i\rangle\langle j|) = \text{tr}(|i\rangle\langle j|) = \delta_{ij}} = \frac{1}{d} \mathbb{1}_B.$$

$\Rightarrow \sigma_{AB} = \hat{X}(\varepsilon) \in \mathcal{F}$ for $\varepsilon \in \mathcal{C}$. ✓

(iv) Let $\sigma_{AB} \in \mathcal{F}$. Write $\sigma_{AB} = \sum_k |\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|$
↑ unnormalized.

We have $|\tilde{\psi}_k\rangle = \sum_{j'} \omega_k^{ij'} |j\rangle|i\rangle = \frac{1}{\Omega} \sum_i (\pi_k \otimes \mathbb{1}) |i\rangle|i\rangle$
 $= \underline{\underline{(\pi_k \otimes \mathbb{1}) |\Omega\rangle}}$

$\Rightarrow \sigma_{AB} = \sum_k (\pi_k \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| (\pi_k \otimes \mathbb{1})^\dagger$, and

$$\begin{aligned} \underline{\underline{\hat{Y}(\sigma_{AB})}}(p) &= d \text{tr}_B[\sigma_{AB} \cdot (\mathbb{1} \otimes p^T)] \\ &= d \text{tr}_B\left[\left(\sum_k (\pi_k \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| (\pi_k \otimes \mathbb{1})^\dagger\right) (\mathbb{1} \otimes p^T)\right] \\ &= d \sum_k \pi_k \underbrace{\text{tr}_B[|\Omega\rangle\langle\Omega| \cdot (\mathbb{1} \otimes p^T)]}_{= \frac{1}{d} \sum_{i,j} |i\rangle\langle j|_A \text{tr}_B[|i\rangle\langle j|_B p^T] = \frac{1}{d} p} \pi_k^\dagger \\ &= \underline{\underline{\sum_k \pi_k p \pi_k^\dagger}}. \end{aligned}$$

$$\text{and } \frac{1}{d} \mathbb{1} = \text{tr}_A \sigma_{AB} = \text{tr}_A \left[\sum_k (\pi_k \otimes \mathbb{1}) |\mathcal{R}\chi\rangle \langle \mathcal{R}\chi| (\pi_k \otimes \mathbb{1})^\dagger \right] \quad (40)$$

$$= \sum_k \text{tr}_A \left[(\pi_k^\dagger \pi_k \otimes \mathbb{1}) |\mathcal{R}\chi\rangle \langle \mathcal{R}\chi| \right]$$

$$= \frac{1}{d} \sum_{ijk} \text{tr} (\pi_k^\dagger \pi_k |i\rangle \langle j|) \otimes |i\rangle \langle j|$$

$$\Rightarrow \langle j | \pi_k^\dagger \pi_k | i \rangle = \text{tr} (\pi_k^\dagger \pi_k |i\rangle \langle j|) = \delta_{ij}$$

$$\Rightarrow \sum_k \pi_k^\dagger \pi_k = \mathbb{1},$$

$$\Rightarrow \hat{\gamma}(\sigma_{AB}) \in \mathcal{L} \text{ for } \sigma_{AB} \in \mathcal{S}. \quad \checkmark$$

Note: The isomorphism still holds if we drop trace preserving and $\text{tr}_A \sigma_{AB} = \frac{1}{d} \mathbb{1}$, respectively.

Corollary (from (iv)): All CPTP maps are of Kraus form (and can thus be realized w/ ancilla + unitary + tracing).

7. Axioms ("mixed version")

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- States are linear operators with

$$\rho \geq 0$$
$$\text{tr} \rho = 1.$$

- Evolution is completely positive trace preserving (CPTP) maps

$$E(\rho) = \sum \Pi_u \rho \Pi_u^\dagger \quad \text{with} \quad \sum \Pi_u^\dagger \Pi_u = \mathbb{1}.$$

- Measurements act as

$$\rho \mapsto f_u = \frac{\Pi_u \rho \Pi_u^\dagger}{\text{tr}(\Pi_u \rho \Pi_u^\dagger)},$$

with prob. $p_u = \text{tr}(\Pi_u^\dagger \Pi_u \rho)$ and $\sum \Pi_u^\dagger \Pi_u = \mathbb{1}$.

IV, Entanglement

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1. Introduction

Consider a bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

$|\psi\rangle$ is a product state if it can be written as

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle.$$

If $|\psi\rangle$ cannot be written in this form, we say

$|\psi\rangle$ is entangled.

Characterization:

- Schmidt coefficients $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$

product state: $\vec{\lambda} = (\lambda_1, 0, \dots, 0)$

entangled state: $\vec{\lambda} = (\lambda_1, \lambda_2 \neq 0, \dots)$

- Reduced density matrix:

product state: $\rho_A = \text{tr}_B |\psi\rangle\langle\psi| = |\phi_A\rangle\langle\phi_A|$

$$\rho_B = |\phi_B\rangle\langle\phi_B|.$$

\Rightarrow reduced state is pure.

And conversely:

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ρ_A pure \Rightarrow Schmidt coeffs $(1, 0, \dots, 0)$

$\Rightarrow |\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$ & $|\chi_A\rangle, |\chi_B\rangle$ determined by ρ_A, ρ_B .

Measured by purity: $\text{tr } \rho_A^2 = 1$ for pure ρ_A s
or some entropies of Schmidt coeffs.

Entangled state:

ρ_A mixed $\Rightarrow \text{tr } \rho_A^2 < 1$.

Entangled states "different":

- Cannot describe parts independently
- meas. outcomes correlated
- will see: suitable for non-trivial tasks.

Aims of study of entanglement:

- How non-classical are entangled states?
- What can we do with them? ("resource")
- How can we quantify amount of entanglement?
- How can we transform/manipulate entanglement?
- What about entanglement of mixed states?