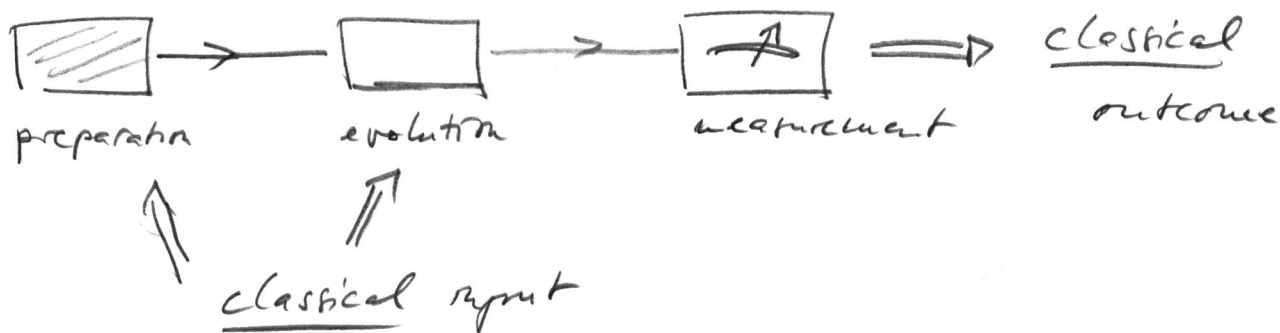


## II. The formalism: States, measurements, evolution

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### 1. Pure states, unitary evolution, projective measurements

Q. II. setup:



Q. II. system → Hilbert space  $\mathcal{H} \cong \mathbb{C}^d$   
(Q. I up: typ. finite dim. H.S.)

State of system: vector  $|\psi\rangle \in \mathcal{H}$  with  $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$ .  
(more precisely: rays  $|\psi\rangle \sim e^{i\phi}|\psi\rangle$ )

Use ket-bra notation:

$|\psi\rangle \in \mathbb{C}^d$ : column vector

$\langle\psi| = (|\psi\rangle)^\dagger$ : row vector

$\langle w|v\rangle$ : scalar product ( $\vec{w}^\dagger \vec{v} = \vec{w} \cdot \vec{v}$ )

## Basis notation:

"Computational basis"  $|0\rangle, |1\rangle, \dots, |d-1\rangle$  of  $\mathbb{C}^d$

$$|k\rangle \triangleq e_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{'th position}$$

$$|v\rangle = \sum_{k=0}^{d-1} v_k |k\rangle = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix}$$

## Linear operations:

$\Pi: \mathbb{C}^d \rightarrow \mathbb{C}^d$  is linear,

$$\Pi(\alpha|v\rangle + \beta|w\rangle) = \alpha\Pi(|v\rangle) + \beta\Pi(|w\rangle)$$

Write  $\Pi|v\rangle \equiv \Pi(|v\rangle)$ .

## Matrix notation / expansion:

$$\begin{aligned} \Pi &= \left( \sum_{i=0}^{d-1} |i\rangle\langle i| \right) \Pi \left( \sum_{j=0}^{d-1} |j\rangle\langle j| \right) = \\ &= \sum_{ij=0}^{d-1} \Pi_{ij} |i\rangle\langle j| = \begin{pmatrix} \Pi_{00} & \Pi_{01} & \dots \\ \Pi_{10} & & \\ \vdots & & \\ & & \Pi_{d-1,d-1} \end{pmatrix} \end{aligned}$$

with  $\Pi_{ij} = \langle i | \Pi | j \rangle$ .

## (i) Preparation:

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Prepares known initial state  $|\phi\rangle \in \mathbb{C}^d$ .

## (ii) Evolution:

Evolution = unitary transformation  $U: \mathbb{C}^d \rightarrow \mathbb{C}^d$ ;

$$|\phi\rangle \mapsto U|\phi\rangle.$$

$$U \text{ unitary} \iff U^\dagger U = U U^\dagger = I \leftarrow \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\left( \text{or: } \sum_j (U^\dagger)_{ij} U_{jk} = \sum_j \overline{U_{ji}} U_{jk} = \delta_{ik} \right)$$

(Note 1:  $\langle \phi | U^\dagger U | \phi \rangle = \langle \phi | \phi \rangle = 1 \Rightarrow$  norm preserved,  
if and only if  $U$  unitary.)

(Note 2:  $U$  can in part be generated by time evolution w/ Hamiltonian.)

## (iii) Measurement:

Observable quantities  $\equiv$  Hermitian operator  $A = A^\dagger$

Eigenvalue decomposition:

$$A = \sum_a a_n E_n ; \quad E_n^2 = E_n = E_n^\dagger \text{ projector onto eigenspace (e.g., } E_n = |\psi_n\rangle\langle\psi_n|)$$

Measurement of  $A$  in state  $|\phi\rangle$ :

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$$\text{Outcome } a_n \text{ w/ prob. } p_n = \langle \phi | E_n | \phi \rangle = \|E_n | \phi \rangle\|^2 \\ (= |\langle \psi_n | \phi \rangle|^2)$$

$$(\text{Note: } \sum p_n = \langle \phi | \underbrace{\sum E_n}_{=1} | \phi \rangle = \langle \phi | \phi \rangle = 1)$$

State after meas.:

$$|\phi_n\rangle = \frac{E_n | \phi \rangle}{\|E_n | \phi \rangle\|}$$

Expectation value:

$$\langle \phi | A | \phi \rangle = \sum a_n \langle \phi | E_n | \phi \rangle$$

## 2. Composite systems

Consider system w/ two separate parts ("subsystems")

$A$  (= Alice) and  $B$  (= Bob).



$\Rightarrow$  Joint system: Hilbert space  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ . (13)

What is general form of  $|\phi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ ?

$|i\rangle_A$  basis of  $\mathcal{H}_A = \mathbb{C}^{d_A}$

$|j\rangle_B$  basis of  $\mathcal{H}_B = \mathbb{C}^{d_B}$

$\Rightarrow |i\rangle_A \otimes |j\rangle_B = |i\rangle_A |j\rangle_B = |ij\rangle_{AB} = |j\rangle_{AB} = |ij\rangle$

is a basis of  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \cong \mathbb{C}^{d_A d_B}$ ,

$i=0, \dots, d_A-1; j=0, \dots, d_B-1$ .

General state  $|\phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ :

$$|\phi\rangle_{AB} = \sum_{\substack{i=0, \dots, d_A-1 \\ j=0, \dots, d_B-1}} c_{ij} |i\rangle_A |j\rangle_B : d_A \cdot d_B \text{-dim. vector } (c_{ij})$$

What if A acts w/  $\mathcal{U}_A$  on her system, and/or B w/  $\mathcal{U}_B$  on his?

(Note:  $\mathcal{U}_A, \mathcal{U}_B$  could be unitaries, measurements ( $E_u$ ),

or "doing nothing",  $\mathcal{U}_B = \mathbb{1}_B$ .)

Consider first  $|\phi\rangle_{AB} = |i\rangle_A \otimes |j\rangle_B$ .

Action of  $A$  should only change her system (as if  $B$  wasn't there):

$$|i\rangle_A \mapsto \pi_A |i\rangle_A, \quad |i\rangle_A \otimes |j\rangle_B \mapsto (\pi_A |i\rangle_A) \otimes |j\rangle_B$$

Same for Bob, formally:

$$\begin{aligned} |i\rangle_A \otimes |j\rangle_B &\mapsto \pi_A |i\rangle_A \otimes N_B |j\rangle_B \\ &\equiv (\pi_A \otimes N_B) |i\rangle_A \otimes |j\rangle_B \end{aligned}$$

Linearity:

$$|\phi\rangle_{AB} \mapsto (\pi_A \otimes N_B) |\phi\rangle_{AB}$$

Matrix elements:

$$\langle i_A i_B | \pi_A \otimes N_B | j_A j_B \rangle = \langle i_A | \pi_A | j_A \rangle \langle i_B | N_B | j_B \rangle$$

$$(\pi_A \otimes N_B)_{(i_A i_B), (j_A j_B)} = (\pi_A)_{i_A j_A} \cdot (N_B)_{i_B j_B}$$

$$\pi_A \otimes N_B = \begin{pmatrix} (\pi_A)_{00} \cdot N_B & (\pi_A)_{01} \cdot N_B & \dots \\ (\pi_A)_{10} \cdot N_B & \dots & \dots \\ \vdots & & \end{pmatrix}$$

## Examples:

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Qubit:  $\mathcal{H} = \mathbb{C}^2$ ,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle; \quad |\alpha|^2 + |\beta|^2 = 1$$

$$\text{Observable } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{|0\rangle\langle 0|}_{E_0} - \underbrace{|1\rangle\langle 1|}_{E_1}$$

$a_0 = +1 \quad a_1 = -1$

Measurement:

outcome  $a_0 = +1$  w/ prob.  $\langle\psi|E_0|\psi\rangle = |\alpha|^2$

outcome  $a_1 = -1$  w/ prob.  $\langle\psi|E_1|\psi\rangle = |\beta|^2$

$$\text{Observable } X = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{|+\rangle\langle +|}_{E_+} - \underbrace{|-\rangle\langle -|}_{E_-}$$

$$\text{with } |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

Measurement: outcomes  $\pm$  w/ prob.  $|\langle\pm|\alpha|0\rangle + \beta|1\rangle|^2 = \frac{|\alpha \pm \beta|^2}{2}$

Evolution:  $U = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  "Hadamard gate"

$$U|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\alpha|0\rangle + \beta|1\rangle)$$

$$= \frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle$$

Meas. in 2-basis  $\{|0\rangle, |1\rangle\}$

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outcome 0 w/ prob.  $\frac{|x+|^2}{2}$

outcome 1 w/ prob.  $\frac{|x-|^2}{2}$

H transfers betw. X and Z bases.

In fact,  $H = \frac{1}{\sqrt{2}}(|+X\rangle + |-X\rangle) = \frac{1}{\sqrt{2}}(|0X+1\rangle + |1X-1\rangle) = H^T$

Measurement on bipartite state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

Alice & Bob measure Z:

project onto  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$$\Rightarrow P_{01} = P_{10} = \frac{1}{2}; P_{00} = P_{11} = 0$$

Alice & Bob measure X:

project onto  $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ :

$$|\langle ++ | \psi \rangle|^2 = 0$$

$$|\langle +- | \psi \rangle|^2 = \left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$|\langle -+ | \psi \rangle|^2 = \dots = \frac{1}{2}$$

$$|\langle -- | \psi \rangle|^2 = 0$$



(using  $\langle +|0\rangle = \langle +|1\rangle = \langle -|0\rangle = -\langle -|1\rangle = \frac{1}{\sqrt{2}}$ )

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$\Rightarrow$  perfect anti-correlation

(in fact, in any basis  $\rightarrow$  Homework!)

Alice meas  $X$ , Bob meas  $Z$ :

$$|\langle +0|4\rangle|^2 = \left|-\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$|\langle +1|4\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$|\langle -0|4\rangle|^2 = \dots = \frac{1}{4}$$

$$|\langle -1|4\rangle|^2 = \dots = \frac{1}{4}$$

Outcomes for A & B are separately completely random,  
but outcomes in the same basis are perfectly anti-corr.