Lecture "Quantum Information" WS 19/20 — Exercise Sheet #3

Problem 1: Dense coding.
Dense coding can be seen as the inverse protocol to teleportation. As in teleportation, Alice and Bob have free entanglement – Bell states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ – at their disposal, but now, they want to use it to transmit classical information by sending quantum states as efficiently as possible, i.e., they want to transmit the maximum amount of classical information per qubit sent. Clearly, by encoding one classical bit into one qubit (e.g., as $|0\rangle$ and $|1\rangle$), they can transmit one classical bit per quantum bit sent. The goal is thus to do better.

1. Show that by acting on her part of $|\Phi^+\rangle$, Alice can transform the shared Bell state $|\Phi^+\rangle$ into any other Bell state.
2. Use this to set up a protocol where Alice can transmit two classical bits by sending only one quantum bit, by using the pre-shared Bell states. This protocol is called *dense coding*, or sometimes *super-dense coding*.
3. Use dense coding, together with teleportation, to show that both protocols are optimal given that shared entanglement is free – this is, we cannot send more classical bits per qubit transmitted, and teleportation of a qubit requires at least two classical bits to be sent (even if we use more complicated protocols sending larger amounts of data at once).

Problem 2: CHSH inequality I: Local hidden variable and quantum mechanics.
Here we will derive another of Bell inequalities, the so-called CHSH inequality. Consider the following scenario: Alice and Bob are each given an input with value (measurement setting) either 0 or 1. We will call their input variables i and j respectively. Without communication between them, they each produce output which is either value -1 or $+1$, which may depend on the input. We will call the output variables a and b respectively. There is also a referee R, who distributes some information to Alice and Bob which they can use to produce their output.

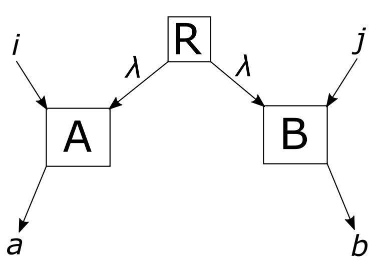

Figure 1: CHSH scenario

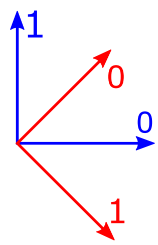

Figure 2: Axis along which Alice (blue) and Bob (red) measure spin

Let us define the quantity $\langle C\rangle$:

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle.$$

Here, $\langle a_i b_j\rangle$ is the expected value of the product of the output variables, given some input i, j. The distribution of the outcomes is described by some joint conditional probability distribution $P(a, b|i, j)$ [i.e., $\sum_{a,b} P(a, b|i, j) = 1$], so

$$\langle a_i b_j\rangle = \sum_{a,b} a\, b\, P(a, b|i, j)\ .$$

1. Classical scenarios can be described by a local hidden variable (LHV) distribution:

   $$P(a, b|i, j) = \sum_{\lambda} p_\lambda P^A_\lambda(a|i) P^B_\lambda(b|j)\ .$$

   Here λ is a variable distributed to Alice and Bob by the referee with a probability distribution p_λ (i.e. $\sum_\lambda p_\lambda = 1$). Use this to derive the bound $|\langle C\rangle| \leq 2$. This should be done by explicitly using the form of $P(a, b|i, j)$, not by making any intuitive assumptions about LHV distributions.

2. For a possible quantum scenario, assume that the referee gives Alice and Bob each one qubit of the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. For Alice, the input determines whether she'll measure the spin along the x axis or the y axis. For Bob, it determines whether he'll measure the spin along the x+y axis or the y-x axis (see Figure 2).

   (a) Determine the observables $\vec{\sigma}_0^A, \vec{\sigma}_1^A, \vec{\sigma}_0^B, \vec{\sigma}_1^B$ corresponding to Alice and Bob doing their measurments along the axes described above.

   (b) The expected value of $\langle a_i b_j \rangle$ can be calculated as

   $$\langle a_i b_j \rangle = \langle \Psi^- | \vec{\sigma}_i^A \vec{\sigma}_j^B | \Psi^- \rangle.$$

   Using that, calculate $\langle C \rangle$ for this particular scenario.

Problem 3: CHSH inequality II: Tsirelson's bound. Here we will again consider the quantum CHSH scenario, but in a more general point of view. Tsirelson's inequality bounds the largest possible violation of the CHSH inequality in quantum mechanics (namely $2\sqrt{2}$). To this end, let A_0, A_1, B_0, B_1 be Hermitian operators with eigenvalues ± 1, so that

$$A_0^2 = A_1^2 = B_0^2 = B_1^2 = \mathbb{1} \ .$$

Here, A_0 and A_1 describe the two measurements of Alice, and B_0 and B_1 those of Bob. That means that operators A_i act as identity on Bob's system and vice versa. Therefore Alice's and Bob's measurements commute, i.e. $[A_i, B_j] = 0$ for all $i, j = 0, 1$. Define the operator

$$C = A_0 B_0 + A_1 B_0 + A_0 B_1 - A_1 B_1 \ .$$

1. Determine C^2.
2. The *operator norm* of a bounded operator M is defined by

   $$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ ,$$

   that is, the operator norm of M is the maximum eigenvalue of $\sqrt{M^\dagger M}$. Verify that this norm has the properties

   $$\|MN\| \leq \|M\| \|N\| \ ,$$
   $$\|M + N\| \leq \|M\| + \|N\| \ .$$

3. Using the rules derived above, find an upper bound on the operator norm $\|C^2\|$.
4. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
5. Explain how this gives a bound on the expected value $\langle C \rangle$. This is known as Tsirelson's bound.

Problem 4: LOCC protocols.
Suppose $|\psi\rangle$ can be transformed to $|\phi\rangle$ by LOCC. A general LOCC protocol can involve an arbitrary number of rounds of measurement and classical communication. In this problem, we will show that any LOCC protocol can be realized in a single round with only one-way communication, i.e., a protocol involving just the following steps: Alice performs a single measurement described by measurement operators K_j, sends the result j to Bob, and Bob performs a unitary operation U_j on his system.
The idea is to show that the effect of any measurement which Bob can do can be simulated by Alice (with one small caveat) so all Bob's actions can actually be replaced by actions by Alice.

1. First, suppose that Bob performs a measurement with operators $M_j = \sum_{kl} M_{j,kl} |k\rangle_B \langle l|_B$ on a pure state $|\psi\rangle_{AB} = \sum_l \lambda_l |l\rangle_A |l\rangle_B$, with the resulting state denoted as $|\psi_j\rangle$. Now suppose that Alice performs a measurement with operators $N_j = \sum_{kl} M_{j,kl} |k\rangle_A \langle l|_A$ on a pure state $|\psi\rangle$, with resulting state denoted as $|\phi_j\rangle$. Show that there exist unitaries U_j on system A and V_j on system B such that $|\psi_j\rangle = (U_j \otimes V_j)|\phi_j\rangle$. *Hint:* Don't try to find the unitary matrices, just try to prove that they exist.
2. Use this to explain how any multi-round protocol can be implemented with one measurement done by Alice followed by a unitary operation done by Bob which depends on Alice's outcome.