

# Problem # 01

## Decay of Entanglement

$$\textcircled{1} \rho(t) = \begin{pmatrix} P_+ & 0 & 0 & e^{-t/\tau_2} \\ 0 & P_- & 0 & 0 \\ 0 & 0 & P_- & 0 \\ e^{-t/\tau_2} & 0 & 0 & P_+ \end{pmatrix}$$

$$\textcircled{2} \rho(t)^{TB} = P_+ |00\rangle\langle 00| + P_- |01\rangle\langle 01| + P_- |10\rangle\langle 10| + P_+ |11\rangle\langle 11| \\ + \frac{e^{-t/\tau_2}}{2} |01\rangle\langle 10| + \frac{e^{-t/\tau_2}}{2} |10\rangle\langle 01|$$

$$2 \cdot \begin{pmatrix} P_+ & 0 & 0 & 0 \\ 0 & P_- & e^{t/\tau_2}/2 & 0 \\ 0 & e^{t/\tau_2}/2 & P_- & 0 \\ 0 & 0 & 0 & P_+ \end{pmatrix}$$

$$\textcircled{3} \lambda_1 = \lambda_2 = P_+$$

$$\lambda_3 = P_- + \frac{1}{2} e^{-t/\tau_2} = \frac{1}{4} - \frac{1}{4} e^{-t/\tau_2} + \frac{1}{2} e^{-t/\tau_2}$$

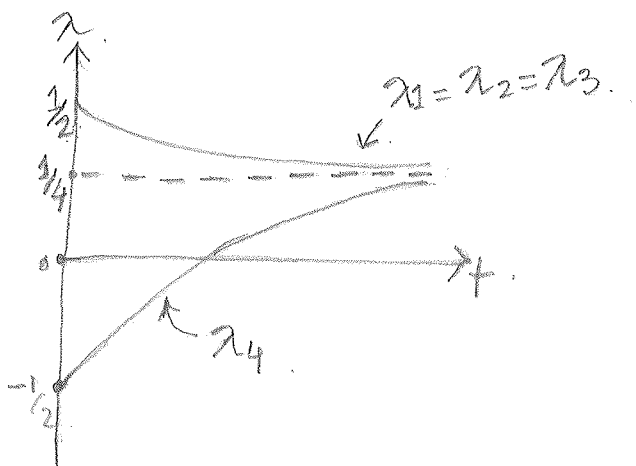
$$\lambda_4 = P_- - \frac{1}{2} e^{-t/\tau_2} = \frac{1}{4} - \frac{1}{4} e^{-t/\tau_2} - \frac{1}{2} e^{-t/\tau_2}$$

$$\textcircled{4} \tau_1 = \tau_2 = 1$$

$$\lambda_1 = \lambda_2 = \frac{1}{4} + e^{-t/4}$$

$$\lambda_3 = \frac{1}{4} + e^{-t/4}$$

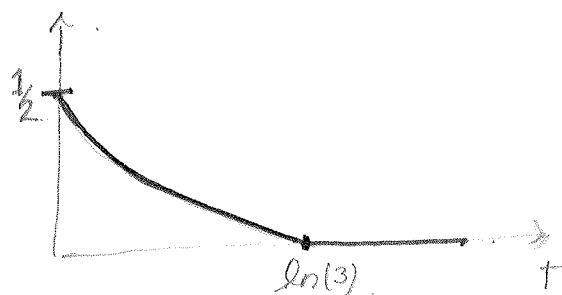
$$\lambda_4 = \frac{1}{4} - \frac{3}{4} e^{-t/4}$$



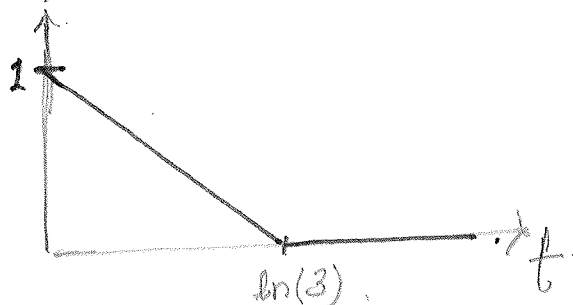
$$N(\rho(t)) = \frac{1}{2} (\|e^{Tt}\|_1 - 1)$$

$$\|e^{Tt}\|_1 = \frac{3}{4} |1 + e^{-t}| + |\frac{1}{4} - \frac{3}{4} e^{-t}|$$

$$N(\rho(t)) = \frac{1}{2} \left( \frac{3}{4} |1 + e^{-t}| + |\frac{1}{4} - \frac{3}{4} e^{-t}| - 1 \right)$$



$$E_{N1}(\rho(t)) = \log_2 \left( \frac{3}{4} |1 + e^{-t}| + |\frac{1}{4} - \frac{3}{4} e^{-t}| \right)$$



⑤ At  $t=0$  PPT (Positive Partial Transpose)

$$n_4(0) < 0 \Rightarrow \rho(t)^{TB} \neq 0$$

$\Rightarrow \rho(0)$  is Entangled.

Since PPT Criterion is necessary and sufficient 202 and 203 cases.

$$n_4(t) = \frac{1}{4} - \frac{3}{4} e^{-t} \geq 0$$

$$\Rightarrow t \geq \ln(3)$$

$\Rightarrow$  state is separable for  $t \geq \ln(3)$ .

# Problem # 02.

## Bell inequalities and witnesses.

- ① For every separable state  $\rho$ ,  
 $|\text{tr}(C\rho)| \leq 2$ . (CHSH inequality)  $\rightarrow$  (\*)  
 $w$  is an entanglement witness if  
 $\text{tr}(w\rho) \geq 0$ .  
separable states

Let  $w = 2I - C$

$$\Rightarrow \text{tr}(w\rho) \stackrel{(*)}{\geq} 2 \text{tr}(\rho) - \text{tr}(C\rho)$$

②  $\text{tr}(w\rho(\lambda)) = \text{tr}(2I\rho(\lambda)) - \text{tr}(C\rho(\lambda))$   
 $= 2 - \text{tr}(C\rho(\lambda))$

$\rho(\lambda) = \lambda |\psi\rangle\langle\psi| + \frac{(1-\lambda)}{4} I$  (Werner state).

$$\text{tr}(w\rho(\lambda)) = 2 - \lambda \text{tr}(C|\psi\rangle\langle\psi|) - \frac{(1-\lambda)}{4} \text{tr}(C)$$

made from Pauli's

$$= 2 - \lambda \langle\psi|C|\psi\rangle$$

But  $C = \sqrt{2} (\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)$

$u_0 = (1, 0, 0)$   
 $u_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$   
 $u_2 = (0, 0, -1)$   
 $u_3 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

$$\Rightarrow \text{tr}(w\rho(\lambda))$$

$$= 2 - \frac{\lambda\sqrt{2}}{2} (\langle 01| - \langle 10|) (|10\rangle - |01\rangle - |01\rangle + |10\rangle)$$

$$= 2 - \frac{\lambda 2}{\sqrt{2}} (\langle 01| - \langle 10|) (|10\rangle - |01\rangle)$$

$$\text{tr}(w \rho(\alpha)) = 2 + 2\sqrt{2}\alpha$$

State is entangled if  $\text{tr}(w \rho(\alpha)) < 0$

$$\Rightarrow 2 + 2\sqrt{2}\alpha < 0$$

$$\boxed{\alpha < -1/\sqrt{2}}$$

Eigenvalues of  $\rho(\alpha)$  are.

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1-\alpha}{4}, \quad \alpha_0 = \alpha + \frac{1-\alpha}{4} = \frac{3\alpha+1}{4}$$

For  $\rho(\alpha)$  to be a density matrix

$$\frac{1-\alpha}{4} \geq 0 \quad \text{and} \quad \frac{3\alpha+1}{4} \geq 0$$

$$\Rightarrow \alpha \leq 1 \quad \text{and} \quad \alpha \geq -1/3$$

$$\alpha \in [-1/3, 1]$$

Since  $-1/\sqrt{2} < -1/3 \Rightarrow w$  cannot detect entanglement in  $\rho(\alpha)$ .

$F$  can detect entanglement in Werner state

### Problem #03

#### Witnesses and Reduction Criterion

$$\textcircled{1} W = \mathbb{I} - d|\Omega\rangle\langle\Omega|, \quad \rho = \sum_i p_i e^{i_A} \otimes e^{i_B}$$

$$\text{tr}(W\rho) = \text{tr}(\rho) - d \text{tr}(|\Omega\rangle\langle\Omega|\rho)$$

$$= 1 - d \langle\Omega|\rho|\Omega\rangle$$

$$= 1 - \frac{d}{d} \sum_{i,j,k} p_i \langle jj| e^{i_A} \otimes e^{i_B} |kk\rangle$$

$$= 1 - \sum_{i,j,k} p_i \langle j| e^{i_A} |k\rangle \langle j| e^{i_B} |k\rangle$$

$$= 1 - \sum_i p_i \text{tr}(e^{i_{AT}} e^{i_B})$$

$$\geq 1 - \sum_i p_i \|e^{i_{AT}}\|_1 \|e^{i_B}\|_\infty$$

$$\geq 0$$

$$\textcircled{2} \rho_{\text{iso}}(\lambda) = \frac{\lambda}{d^2} \mathbb{I} + (1-\lambda)|\Omega\rangle\langle\Omega|$$

$$\rho_{\text{iso}}(\lambda)|\Omega\rangle = \left(\frac{\lambda}{d^2} + (1-\lambda)\right)|\Omega\rangle = \frac{\lambda(1-d^2) + d^2}{d^2} |\Omega\rangle \leftarrow |\psi_0\rangle$$

Let  $|\psi_i\rangle \perp |\Omega\rangle$  for  $i=1, 2, \dots, d^2-1$

$$\rho_{\text{iso}}|\psi_i\rangle = \frac{\lambda}{d^2} |\psi_i\rangle$$

$$\Rightarrow \rho_{\text{iso}}(\lambda) > 0 \text{ iff } \frac{\lambda}{d^2} \geq 0 \text{ and } \frac{\lambda(1-d^2) + d^2}{d^2} \geq 0$$

$$\Leftrightarrow \lambda \geq 0 \text{ and } \lambda \leq \frac{d^2}{d^2-1}$$

$$\text{so } \lambda \in [0, \frac{d^2}{d^2-1}]$$

$$\text{tr}(W\rho_{\text{iso}}(\lambda)) = \text{tr}((\mathbb{I} - d|\Omega\rangle\langle\Omega|)\rho_{\text{iso}}(\lambda))$$

$$= 1 - d \langle\Omega|\rho_{\text{iso}}(\lambda)|\Omega\rangle = 1 - \frac{d}{d^2} \lambda - d(1-\lambda)$$

$$= (d - d^2 + 2(d^2 - 1))/d < 0$$

$$\text{iff } 2(d-1) < d^2 - d$$

$$2 < \frac{d^2 - d}{d^2 - 1}$$

$$\Rightarrow \omega \text{ detect } \rho(\lambda) \text{ is for } \lambda \in [0, \frac{d}{d+1}]$$

$$\textcircled{3} \quad d=2, \omega = \mathbb{I} - 2|\phi^+\rangle\langle\phi^+|, \rho = |\psi^-\rangle\langle\psi^-|$$

$$\begin{aligned} \text{tr}_B(\omega|\psi^-\rangle\langle\psi^-|) &= \langle\psi^-|\omega|\psi^-\rangle \\ &= \langle\psi^-|(\mathbb{I} - 2|\phi^+\rangle\langle\phi^+|)|\psi^-\rangle \\ &= 1 > 0 \end{aligned}$$

$\omega$  does not detect if  $\rho = |\psi^-\rangle\langle\psi^-|$  is entangled.

$$\textcircled{4} \quad \Lambda(X) = \text{tr}_A(\omega(X_A \otimes I_B))^T$$

$$= \text{tr}_A((\mathbb{I} - d|\Omega\rangle\langle\Omega|)(X_A \otimes I_B))^T$$

$$= \text{tr}_A(X_A \otimes I_B)^T - \text{tr}_A\left(\sum_{i,j} |ii\rangle\langle jj| (X_A \otimes I_B)\right)^T$$

$$= \mathbb{I} \cdot \text{tr}(X) - \left(\sum_{i,j} \langle i'|i\rangle \langle j|X|i'\rangle |i\rangle\langle j|\right)^T$$

$$= \mathbb{I} \text{tr}(X) - \left(\sum_{i,j} \langle j|X|i\rangle |i\rangle\langle j|\right)^T$$

$$= \mathbb{I} \text{tr}(X) - \sum_{i,j} \langle j|X|i\rangle |j\rangle\langle i|$$

$$\Lambda(X) = \mathbb{I} \text{tr}(X) - \tau(X)$$

$X \geq 0 \Rightarrow \{ \lambda_i, |\psi_i\rangle \}_{\lambda_i \geq 0}$  be the eigenvalue decomposition

$$\Rightarrow \Lambda(X) = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right) \left(\sum_j \lambda_j\right) - \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$= \sum_i |4_i\rangle \langle 4_i| \left( \sum_j \lambda_j - \lambda_i \right)$$

$$\geq 0$$

$$\Rightarrow \Lambda(x) \geq 0$$

$$\textcircled{5} \quad (\Lambda \otimes \mathbb{I})(\rho_{\text{iso}}(\lambda)) = (\Lambda \otimes \mathbb{I}) \left( \lambda \frac{\mathbb{I}}{d^2} + (1-\lambda) |\Omega\rangle \langle \Omega| \right)$$

$$= (\Lambda \otimes \mathbb{I}) \left( \lambda \frac{\mathbb{I}_A \otimes \mathbb{I}_B}{d^2} + \frac{(1-\lambda)}{d} \sum_{ij} |i\rangle \langle j| \otimes |i\rangle \langle j| \right)$$

$$= \frac{\lambda}{d^2} \Lambda(\mathbb{I}_A) \otimes \mathbb{I}_B + \frac{(1-\lambda)}{d} \sum_{ij} \left( \Lambda(|i\rangle \langle j|) \otimes |i\rangle \langle j| \right)$$

$$= \frac{\lambda}{d^2} (\mathbb{I}_A + \lambda(\mathbb{I}_A) - \mathbb{I}_A) \otimes \mathbb{I}_B + \frac{(1-\lambda)}{d} \sum_{ij} (\mathbb{I}_A + \lambda(|i\rangle \langle j|) - |i\rangle \langle j|) \otimes |i\rangle \langle j|$$

$$= \frac{\lambda}{d^2} (d-1) \mathbb{I} + \frac{(1-\lambda)}{d} \sum_{ij} (\delta_{ij} \mathbb{I}_A \otimes |i\rangle \langle j| - |i\rangle \langle j| \otimes |i\rangle \langle j|)$$

$$= \frac{\lambda}{d^2} (d-1) \mathbb{I} + \frac{(1-\lambda)}{d} \mathbb{I} - (1-\lambda) |\Omega\rangle \langle \Omega|$$

$$\text{Eigenvalues of } (\Lambda \otimes \mathbb{I})(\rho_{\text{iso}}(\lambda)) = \left\{ \frac{\lambda(d-1) + d(1-\lambda)}{d^2}, \frac{\lambda(d-1) + d(1-\lambda) - (1-\lambda)d^2}{d^2} \right\}$$

$$\neq 0 \text{ iff } \lambda(d-1) + d(1-\lambda) < 0$$

$$\text{or } \lambda(d-1) + d(1-\lambda) - (1-\lambda)d^2 < 0$$

$$\Leftrightarrow \lambda > d \text{ or } \lambda < \frac{d^2 - d}{d^2 - 1}$$

$$\text{But for } \rho_{\text{iso}}(\lambda) \text{ to be } \geq 0 \quad \lambda \in \left[ 0, \frac{d^2}{d^2 - 1} \right]$$

$\Rightarrow$  A detect  $\rho_{\text{iso}}(\lambda)$  is entangled

$$\text{iff } \lambda \in \left[ 0, \frac{d^2 - d}{d^2 - 1} \right]$$

$$\begin{aligned}
 (\Lambda \otimes I)(|\psi^-\rangle\langle\psi^-|) &= \frac{1}{2} \left( \Lambda(|0\rangle\langle 0|) |1\rangle\langle 1| - \Lambda(|0\rangle\langle 1|) |1\rangle\langle 0| \right. \\
 &\quad \left. - \Lambda(|1\rangle\langle 0|) |0\rangle\langle 1| + \Lambda(|1\rangle\langle 1|) |0\rangle\langle 0| \right) \\
 &= \frac{1}{2} \left( |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| \right. \\
 &\quad \left. + |0\rangle\langle 0| \otimes |0\rangle\langle 0| \right)
 \end{aligned}$$

$$\circ \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0 \Rightarrow \Lambda \text{ detect entanglement in } |\psi^-\rangle\langle\psi^-|.$$