1. *(Displacement operator)* For operators A, B such that both A and B commute with their commutator $[A, B] = AB - BA$ the Baker-Campbell-Hausdorff formula states that $\exp(A + B) = \exp(A)\exp(B)\exp(-[A, B]/2)$. Show that the displacement operator $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is equal to $e^{-|\alpha|^2/2}\exp(\alpha a^\dagger)\exp(-\alpha^* a)$.

   Using the above, show that $D(\alpha)^\dagger a D(\alpha) = a + \alpha$.

2. *(Squeezing)* The *field quadrature operators* for a single mode are defined as $X := (a + a^\dagger)/\sqrt{2}$ and $P := (a - ia^\dagger)/\sqrt{2}$. Compute the variance $\Delta X^2 \equiv \mathrm{Var}(X)_\rho = \mathrm{tr}(\rho X^2) - [\mathrm{tr}(\rho X)]^2$ and $\mathrm{Var}(X)_\rho$ in the vacuum state $\rho = |0\rangle\langle 0|$ and show that it saturates the uncertainty relation $\Delta X \Delta P \geq 1/2$ ("minimal uncertainty state"). Show that the variances are unchanged by the unitary operation $D(\alpha)$.

   To see that this is indeed a special property, calculate the variances also in a number state $|n\rangle$.

   Now consider a Hamiltonian $H = \eta(a^2 + (a^\dagger)^2)$. Find the equations of motion for the operators $a, a^\dagger$.

   What is the equation for the variances of X and P? Solve it.

3. *(Classically driven two-level atom)* The Hamiltonian

$$H = \hbar\nu \frac{1}{2}\sigma^z + \hbar\Omega_0 \cos(\omega t)(\sigma^+ + \sigma^-) \quad (1)$$

   is often used to describe the interaction of an atom with a single-mode laser field. ($\sigma^z = |1\rangle\langle 1| - |0\rangle\langle 0|$ and $\sigma^+ = |1\rangle\langle 0| = (\sigma^-)^\dagger$)

   What are the main approximations needed to obtain it from the minimal coupling Hamiltonian discussed in the lecture?

   Find the Heisenberg equations of motion for the Pauli matrices $\sigma^u, u = x, y, z$.

   Simplify them using the rotating wave approximation and solve them in the case of resonance $\omega = \nu$.