1. **Collective coupling of atoms to cavity mode**
One way to obtain strong and readily observable effects in cavity-QED is by coupling to a large number of atoms (the other is using high-Q micro-cavities). The purpose of this exercise is to derive the collective enhancement of the coupling and determine the associated energy level structure, which differs distinctly from the Jaynes-Cummings case discussed in class.
Historically, collective coupling was the first instance for which effects like such as vacuum Rabi splitting were observed in the optical regime[1]. More recently it has received renewed attention as a means for "quantum memory" and "light-matter interface" in quantum information[2].

We consider two-level atoms with ground state $|g\rangle$ and excited state $|e\rangle$ with the Hamiltonian
$$H = H_0 + H_{\rm int},$$
where $H_0 = \frac{1}{2}\hbar\omega_a\sigma_z + \hbar\omega_c(a^\dagger a + \frac{1}{2})$ and the interaction Hamiltonian coupling an atom to the cavity mode a (in the usual approximations) is
$$H_{\rm int} = i\hbar g\left(\sigma_+ a - \sigma_- a^\dagger\right).$$
where $\sigma_- = \sigma_+^\dagger = |g\rangle\langle e|$.

a) Consider a collection of N identical atoms all coupling to the same cavity mode by $H_{\rm int}$ with the same constant g. Then the overall coupling atoms-field is described by
$$H_{\rm int} = i\hbar g\left(\sum_{r=1}^{N}\sigma_-^{(r)}\right)a^\dagger - i\hbar g\left(\sum_{r=1}^{N}\sigma_+^{(r)}\right)a.$$
Show that the collective operators $\left(\sum_{r=1}^{N}\sigma_\pm^{(r)}\right)$ coupling the atoms to the cavity mode are angular momentum ladder operators $J_\pm$. (*Hint:* This can be done, for example, by checking their commutation relations, using that the $\frac{1}{2}\sigma_j^{(r)}$ satisfy the angular momentum relations $[J_k, J_l] = i\epsilon_{klm}J_m$, where ϵ_{klm} is the completely antisymmetric tensor.)

b) According to the rules for the addition of angular momenta, a complete orthonormal basis of the Hilbert space of N spin-1/2 systems is given by the collective angular momentum states ("Dicke states") $|J, M, \alpha\rangle$, where $J(J+1)$ is the eigenvalue of $J_x^2 + J_y^2 + J_z^2$, M the eigenvalue of J_z and α an additional quantum number to lift the degeneracy of J^2 and J_z. In the following, we

[1] Raizen *et al.*, Phys. Rev. Lett. **63**, 437 (1989)
[2] C. Schori *et al.*, Phys. Rev. Lett. **89**, 057903 (2002)

consider the case $J = N/2$ for which there is no additional degeneracy and α can be omitted.[3]

What are the ranges of J, m? Give the states $|J, -J\rangle$ for $J = N/2$ and for $J = 0$ in terms of atomic states.

c) Consider the system prepared in the state $|\psi_0\rangle = \left|\frac{N}{2}, -\frac{N}{2}\right\rangle |1\rangle$ (all atoms in the ground state and one photon in the cavity).
Solve the Schrödinger equation for H with initial condition $|\psi_0\rangle$. (*Hint:* Use products of (Dicke states)$\otimes$(number states) as basis and show that H couples $|\psi_0\rangle$ to only one such state. Here and in the following you can restrict to the resonant case $\omega_a = \omega_c$.)
Compare to the case $N = 1$ discussed in the lecture.

d) These are two special cases of a more general property of H: Show that the operators $X_1 = J_x^2 + J_y^2 + J_z^2$ ("total angular momentum") and $X_2 = J_z + a^\dagger a$ ("excitation number") are conserved by H.
This implies that H does not couple subspaces corresponding to different eigenvalues of these operators. We denote these subspaces by $\mathcal{H}_{J,E}$, where J, E are related to the eigenvalues of X_1, X_2 by $x_1 = J(J+1)$ and $x_2 = E - J$.
By the above, H can be written as a direct sum of Hamiltonians $H_{J,E}$ acting on the subspaces $\mathcal{H}_{J,E}$, i.e., $H = \bigoplus_{(J,E)} H_{J,E}$.

e) Compute the energy eigenvalues of $H_{N/2,n}$ for $n = 0, 1, 2 \ll N$ (here and in the following, you may neglect terms of order $1/N$) and sketch the energy-level scheme of the atoms-cavity system both in the case $g = 0$ (using "bare" states) and for $g \neq 0$ (using "dressed" states). (*Hint:* Use $J_+ |J, M\rangle = \sqrt{(J + M + 1)(J - M)}\, |J, M + 1\rangle$. The 6 eigenstates are $|J, -J\rangle |0\rangle$; $\{|J, -J\rangle |1\rangle \pm i |J, 1 - J\rangle |0\rangle\}$; $\{|J, 2 - J\rangle |0\rangle \pm i\sqrt{2} |J, 1 - J\rangle |1\rangle - |J, -J\rangle |2\rangle, |J, 2 - J\rangle |0\rangle + |J, -J\rangle |2\rangle\}$.)

f) What optical transition energies (i.e. transitions between states of different "excitation number" n) occur?
These lines can only show up in the spontaneous emission spectrum of the system, if initial and final state are connected by J_-, i.e., $\langle f| J_- |i\rangle \neq 0$. Show that the levels considered here produce only two lines in the spectrum. This can be shown to hold even if larger n are considered (as long as $n \ll N$). Compare this with the spectrum obtained in class for the case $N = 1$.

2. **Cavity decay**
We consider a lossy single-mode cavity. The losses can be described by a bilinear

[3] One possible choice for α are the eigenvalues of $\sigma_z^{(r)}$; this makes it easy to find the degeneracy of the subspace (J, M) by combinatorial considerations. However there is a more convenient choice such that the value α is unaffected by the action of $J_\pm$. Then α is conserved by H it can be neglected for our discussion here.

coupling of the cavity mode to the surrounding one-dimensional free field:

$$H = \hbar\omega_c(a^\dagger a + 1/2) + \hbar\int_{\mathbb{R}} \omega(b_\omega^\dagger b_\omega + 1/2)d\omega + i\hbar\int_{\mathbb{R}} \kappa(\omega)(a^\dagger b_\omega - b_\omega^\dagger a). \quad (1)$$

[The integral over positive *and negative* frequencies can be justified by noting that we may shift the physical range $(0,\infty)$ to $(-\Omega,\infty)$ by changing into an interaction picture with a optical (large) frequency Ω of the order of the cavity frequency; ω_c in Eq. (1) is then the shifted cavity frequency and $\ll \Omega$. Since Ω is very large, extending the integral to $\mathbb{R}$ is a good approximation.[4]] In the following we assume a frequency independent "white" coupling $\kappa(\omega) = \gamma/(2\pi)$.

a) Classically, one would expect the amplitude of the field in the cavity to decay exponentially. What is fundamentally wrong with the corresponding "semi-classical" ansatz $a(t) = e^{-\gamma/2t}a(0)$ for the cavity mode operator?

b) Write the Heisenberg equations of motion for a, b_ω. Solve formally for b_ω and insert the result in the equation for a. (*Hint:* use $\int_{\mathbb{R}} e^{-i\omega t}d\omega = 2\pi\delta(t)$.) Identify the decay term and the Langevin force.

c) Verify that the commutation relations of $a, a^\dagger$ are preserved.

3. Consider the Jaynes-Cummings Hamiltonian

$$H = \frac{1}{2}\hbar\omega\sigma^z + \hbar\nu a^\dagger a + \hbar g\left(\sigma^+ a + \sigma^- a^\dagger\right).$$

Find the eigenvalues and eigenvectors. Simplify the expressions in the resonant $\omega = \nu$ and far-detuned $\Delta = \omega - \nu \gg g$ cases. Plot the eigenvalues of H in teh $n \geq 1$-excitation subspace as a function of Δ.

[4] If you don't like this argument, do without it: Using $\int_0^\infty e^{-i\omega t}d\omega = \pi\delta(t) - i\frac{\mathcal{P}}{t}$, where $\frac{\mathcal{P}}{t}$ behaves as $1/t$ except for a small interval around 0, where it vanishes, the expressions become only slightly more complicated due to this last term.