3.1) Displacement operator

(i) Using the Baker-Campbell-Hausdorff formula we obtain

$$e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{|\alpha|^2 [a^\dagger, a]/2} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2}.$$

(ii) To show that $D(\alpha)^\dagger a D(\alpha) = a + \alpha$, we use the equation above.

$$D(\alpha)^\dagger a D(\alpha) = e^{\alpha^* a} e^{-\alpha a^\dagger} e^{|\alpha|^2/2} a \; e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\alpha^* a} e^{-\alpha a^\dagger} a \; e^{\alpha a^\dagger} e^{-\alpha^* a}.$$

The expression $a \; e^{\alpha a^\dagger}$ is now manipulated such that the annihilators are on the right side, while the creation operators move to the left (next to the expression $e^{-\alpha a^\dagger}$). To do so, we expand $e^{\alpha a^\dagger}$ in a series and the use the relation

$$[a, (a^\dagger)^n] = n (a^\dagger)^{n-1}.$$

We obtain

$$\begin{aligned} a \; e^{\alpha a^\dagger} &= a \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \Big(n (a^\dagger)^{n-1} + (a^\dagger)^n a \Big) = e^{\alpha a^\dagger} a + \alpha \sum_{n_0}^{\infty} \frac{\alpha^{n-1}}{(n-1)!} (a^\dagger)^{n-1} \\ &= e^{\alpha a^\dagger} a + \alpha \; e^{\alpha a^\dagger} = e^{\alpha a^\dagger} (a + \alpha), \end{aligned}$$

and therefore

$$D(\alpha)^\dagger a D(\alpha) = e^{\alpha^* a} e^{-\alpha a^\dagger} a \; e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\alpha^* a} e^{-\alpha a^\dagger} e^{\alpha a^\dagger} (a + \alpha) e^{-\alpha^* a} = (a + \alpha).$$

3.2) Squeezing

(i) First the variances ΔX^2 and ΔP^2 in the vacuum state are computed.

$$\begin{aligned} \Delta X^2 &= tr\{\rho X^2\} - [tr\{\rho X\}]^2 = tr\{|0\rangle\langle 0| \frac{1}{2} (a + a^\dagger)(a + a^\dagger)\} - [tr\{|0\rangle\langle 0| \frac{1}{\sqrt{2}} (a + a^\dagger)\}]^2 \\ &= tr\{|0\rangle\langle 0| \frac{1}{2} (aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)\} - 0 = \frac{1}{2} \langle 0| aa^\dagger |0\rangle + \frac{1}{2} \langle 0| a^\dagger a |0\rangle = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \Delta P^2 &= tr\{\rho P^2\} - [tr\{\rho P\}]^2 = tr\{|0\rangle\langle 0| (-\frac{1}{2}) (a - a^\dagger)(a - a^\dagger)\} - [tr\{|0\rangle\langle 0| (\frac{-i}{\sqrt{2}}) (a - a^\dagger)\}]^2 \\ &= tr\{|0\rangle\langle 0| (\frac{-1}{2}) (aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger)\} - 0 = \frac{1}{2} \langle 0| aa^\dagger |0\rangle + \frac{1}{2} \langle 0| a^\dagger a |0\rangle = \frac{1}{2}. \end{aligned}$$

The variances ΔX^2 and ΔP^2 in the vacuum state fulfill the minimum-uncertainty relation $\Delta X^2 \Delta P^2 = 1/4$. The operation $D(\alpha)$ describes only a displacement

$$\begin{aligned} a_D &= D(\alpha)^\dagger a D(\alpha) = a + \alpha, \\ a_D^\dagger &= D(\alpha)^\dagger a^\dagger D(\alpha) = a^\dagger + \alpha^*, \\ X_D &= D(\alpha)^\dagger X D(\alpha) = (a_D + a_D^\dagger)/\sqrt{2} = X + (\alpha + \alpha^*)\sqrt{2}, \\ P_D &= D(\alpha)^\dagger P D(\alpha) = -i(a_D - a_D^\dagger)/\sqrt{2} = P - i(\alpha - \alpha^*)\sqrt{2}. \end{aligned}$$

Therefore it does not change the variances

$$\begin{aligned} \Delta X_D^2 &= \Delta[X + (\alpha + \alpha^*)\sqrt{2}]^2 = \Delta X^2, \\ \Delta P_D^2 &= \Delta[P - i(\alpha - \alpha^*)\sqrt{2}]^2 = \Delta P^2. \end{aligned}$$

(ii) We start with the Heisenberg equations for the operators a and $a^\dagger$

$$\begin{aligned} \partial_t a &= i[H, a] = i\eta[(a^\dagger)^2, a] = -2i\eta a^\dagger, \\ \partial_t a^\dagger &= i[H, a^\dagger] = i\eta[a^2, a^\dagger] = 2i\eta a. \end{aligned}$$

By differenting one of these equations and substituting the other one we obtain

$$\begin{aligned} \partial_t^2 a &= -2i\eta \partial_t a^\dagger = 4\eta^2 a, \\ \partial_t^2 a^\dagger &= 2i\eta \partial_t a = 4\eta^2 a^\dagger, \end{aligned}$$

which is solved by

$$\begin{aligned} a(t) &= a(0)\cosh(2\eta t) - a^\dagger(0)\sinh(2\eta t), && (1) \\ a^\dagger(t) &= a^\dagger(0)\cosh(2\eta t) - a(0)\sinh(2\eta t). && (2) \end{aligned}$$

Now we have a look at the variances of X and P

$$\begin{aligned} \Delta X(t)^2 &= \frac{1}{2}\langle vac|a(t)a(t) + a^\dagger(t)a(t) + a(t)a^\dagger(t) + a^\dagger(t)a^\dagger(t)|vac\rangle \\ &\quad -\frac{1}{2}\langle vac|a(t) + a^\dagger(t)|vac\rangle^2, \\ \Delta P(t)^2 &= \frac{1}{2}\langle vac| - a(t)a(t) + a^\dagger(t)a(t) + a(t)a^\dagger(t) - a^\dagger(t)a^\dagger(t)|vac\rangle \\ &\quad +\frac{1}{2}\langle vac|a(t) - a^\dagger(t)|vac\rangle^2. \end{aligned}$$

It is clear from equations 1 and 2, that $\langle vac|a(t)|vac\rangle = \langle vac|a^\dagger(t)|vac\rangle = 0$ and accordingly $\frac{1}{2}\langle vac|a(t) \pm a^\dagger(t)|vac\rangle^2 = 0$. In order to evaluate the remaining terms, we calculate $\langle vac|a(t)a(t)|vac\rangle$, $\langle vac|a^\dagger(t)a^\dagger(t)|vac\rangle$ and $\langle vac|a(t)a^\dagger(t)+a^\dagger(t)a(t)|vac\rangle$.

$$\begin{aligned}
\langle vac|a(t)a(t)|vac\rangle &= \frac{1}{2}\langle vac|(a(0)\cosh(2\eta t) - a^\dagger(0)\sinh(2\eta t)) \\
&\quad (a(0)\cosh(2\eta t) - a^\dagger(0)\sinh(2\eta t))|vac\rangle \\
&= \langle vac|a^2(0)\cosh(2\eta t)^2 - a(0)a^\dagger(0)\cosh(2\eta t)\sinh(2\eta t) \\
&\quad -a^\dagger(0)a(0)\cosh(2\eta t)\sinh(2\eta t) + (a^\dagger(0))^2\sinh(2\eta t)^2|vac\rangle \\
&= -\cosh(2\eta t)\sinh(2\eta t)\langle vac|a(0)a^\dagger(0) + a^\dagger(0)a(0)|vac\rangle \\
&= -\cosh(2\eta t)\sinh(2\eta t)\langle vac|a(0)a^\dagger(0)|vac\rangle = -\cosh(2\eta t)\sinh(2\eta t)
\end{aligned}$$

Analogously $\langle vac|a^\dagger(t)a^\dagger(t)|vac\rangle = -\cosh(2\eta t)\sinh(2\eta t)$ and

$$\begin{aligned}
\langle vac|a(t)a^\dagger(t) + a^\dagger(t)a(t)|vac\rangle &= \langle vac|(a(0)\cosh(2\eta t) - a^\dagger(0)\sinh(2\eta t)) \\
&\quad (a^\dagger(0)\cosh(2\eta t) - a(0)\sinh(2\eta t)) \\
&\quad +(a^\dagger(0)\cosh(2\eta t) - a(0)\sinh(2\eta t)) \\
&\quad (a(0)\cosh(2\eta t) - a^\dagger(0)\sinh(2\eta t))|vac\rangle \\
&= \cosh(2\eta t)^2 + \sinh(2\eta t)^2.
\end{aligned}$$

By inserting these expressions into the equations for $\Delta X(t)^2$ and $\Delta P(t)^2$ we can infer

$$\begin{aligned}
\Delta X(t)^2 &= \frac{1}{2}(\cosh(2\eta t)^2 + \sinh(2\eta t)^2 - \cosh(2\eta t)\sinh(2\eta t)) = \frac{1}{2}e^{-4\eta t}, \\
\Delta P(t)^2 &= \frac{1}{2}(-\cosh(2\eta t)^2 - \sinh(2\eta t)^2 + \cosh(2\eta t)\sinh(2\eta t)) = \frac{1}{2}e^{4\eta t}.
\end{aligned}$$

The variance of the X quadrature is squeezed, while the variance of P is anti-squeezed, such that the minimum-uncertainty product $\Delta X(t)^2\Delta P(t)^2 = 1/4$ is conserved.