5.1) Collective coupling of atoms to a cavity mode

a) Let $X_q = \frac{1}{2}\sum_r \sigma_q^{(r)}$ $(q = x, y, z)$. The angular momentum operators are characterized by their commutation relations $[J_r, J_s] = i\epsilon_{rst}J_t$ with ϵ_{rst} the totally antisymmetric tensor. The ladder operators are $J_\pm = J_x \pm iJ_y$. $X_\pm$ are given similarly in terms of the $X_{x,y}$: (*The factor* 1/2 *in front of* $\sum \sigma_-^{(r)}$ *in 1a) was a typo!*) Hence if we can show that $[X_r, X_s] = i2\epsilon_{rst}X_t$ the claim follows. We have

$$[X_r, X_s] = \sum_{k,l} \frac{1}{4}[\sigma_r^{(k)}, \sigma_s^{(l)}] = \frac{1}{4}\sum_k i2\epsilon_{rst}\sigma_t^{(k)} = i\epsilon_{rst}X_t.$$

b) Since the X_r are have been shown to be angular momentum operators, For N even, $J \in \{0, 1, \ldots, N/2\}$, for N odd, $J \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{N}{2}\}$. And $m \in \{-J, 1-J, \ldots, J-1, J\}$. To prove that no other the J values than the above can be reached, use induction and the rules for addition of angular momenta: for $N = 1$, the above holds. Assume it also holds for N. Adding a spin-1/2 particle to a spin-J system allows to obtain $J' = J \pm 1/2$ but no more. (To see this, the case of obtaining a state with $(J', J'_z = -J')$: since $J'_z = J_z + S_z$, where S_z is the)

c) Assuming that all atoms couple identically to the mode, the Hamiltonian is and setting $\omega_a = \omega_c = \omega$

$$\hbar\omega J_z + \hbar\omega(a^\dagger a + \frac{1}{2}) + i\hbar g\left(J_+ a - J_- a^\dagger\right)$$

The first part is diagonal in the $|n\rangle\,|J, M, \alpha\rangle$-basis. The interaction part couples $|n\rangle\,|J, m\rangle$ to $|n-1\rangle\,|J, m+1\rangle$ and to $|n+1\rangle\,|J, m-1\rangle$, i.e. a photon can be absorbed to increase J_z by one or it can be emitted into the cavity be reducing J_z by one. The latter process is not possible in a state $|J, -J\rangle$ (and the former not in the case $n = 0$). Therefore, starting in $|1\rangle\,|J, -J\rangle$ limits the dynamics to the two-dimensional subspace $\text{span}\{|1\rangle\,|J, -J\rangle, |0\rangle\,|J, 1-J\rangle\}$. (A special case is a Jaynes-Cummings system with a single spin-1/2 in the ground state and an single photon in the cavity.)

In this subspace, the Hamiltonian is

$$\hbar\begin{pmatrix} \omega - \omega J & -ig\sqrt{2J} \\ ig\sqrt{2J} & \omega(1-J) \end{pmatrix}$$

where $J_\pm\,|J, m\rangle = \sqrt{(J \mp m)(J \pm m + 1)}\,|J, m \pm 1\rangle$ was used for the special case $m = -J$. Thus compared to the single-atom case we have a *collective enhancement* of the coupling by a factor $\sqrt{2J} = \sqrt{N}$.

d) $J_{x,y,z}$ commute with X_1 (see any QM textbook) and so do $a, a^\dagger$ which act on a different system, hence so does H, which contains only $J_r, a, a^\dagger$-operators.

Considering X_2, use $[J_z, J_\pm] = \pm J_\pm$ and $[a^\dagger a, a] = -a$, $[a^\dagger a, a^\dagger] = a^\dagger$. The diagonal part of H clearly commutes with X_2, for the interaction part we have $[H_{\text{int}}, X_2] = iga^\dagger [J_-, J_z] - iga[J_j, J_z] + ig[a^\dagger, a^\dagger a]J_- - ig[a, a^\dagger a]J_+$. Inserting the commutation rules for J_r, a from above shows $[H_{\text{int}}, X_2] = 0$.

e) We have to find eigenvectors (and eigenvalues) of H_{int} in a subspace of given x_1, x_2.

$x_2 = -J$ here we only have $|J, -J\rangle\, |0\rangle$ with H_{int}-eigenvalue 0, i.e. with total energy $-\omega J$.

$x_2 = 1 - J$ spanned by $|J, -J\rangle\, |1\rangle$ and $|J, 1 - J\rangle\, |0\rangle$. Eigenvectors are

$$|J, -J\rangle\, |1\rangle \pm i\, |J, 1 - J\rangle\, |0\rangle\,,$$

which we denote by $|\pm; 1\rangle$. The corresponding eigenvalues are $\omega(1-J) \pm g\sqrt{2J}$.

$x_2 = 2 - J$ is spanned by $\{|J, 2 - J\rangle\, |0\rangle\,, |J, 1 - J\rangle\, |1\rangle\,, |J, -J\rangle\, |2\rangle\}$. On this subspace, we have

$$H_{\text{int}} = g \begin{pmatrix} 0 & i\sqrt{2(2J-1)} & 0 \\ i2\sqrt{J} & 0 & -i\sqrt{2(2J-1)} \\ 0 & -i\sqrt{2}\sqrt{2J} & 0 \end{pmatrix} \approx ig2\sqrt{J} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

if we neglect terms of order $O(1/J)$, which is ok for $J = N/2$ or more generally for $J = O(N)$. The eigenvalues of this matrix are $g2\sqrt{J}\{\pm\sqrt{2}, 0\}$ and the corresponding eigenvectors are

$$\{|J, 2 - J\rangle\, |0\rangle \mp i\sqrt{2}\, |J, 1 - J\rangle\, |1\rangle - |J, -J\rangle\, |2\rangle\,, |J, 2 - J\rangle\, |0\rangle + |J, -J\rangle\, |2\rangle\},$$

which we denote by $|\mp, 2\rangle$ and $|0; 2\rangle$. (*There was a sign wrong in the Hint in part e): the last vector given there is not orthogonal to the previous one an should have a "+" instead of a" -"!*)

f) For fixed J, the level scheme in the bare case consists of equally spaced ($\Delta E = \hbar\omega$), multiply-degenerate levels. For $n \leq 2J + 1$, the level $E = n\hbar\omega$ is n-fold degenerate and spanned by the states $\{|J, l - J\rangle\, |n - l\rangle : l = 0, \ldots, n\}$. For $n > 2J + 1$, the degeneracy is $2J + 1$ and the eigenvectors spanning the subspace are $\{|J, l - J\rangle\, |n - l\rangle : l = 0, \ldots, 2J\}$.

The degeneracy is lifted due to the interaction, e.g., the level $E = \omega$ (spanned by $|J, -J, 1\rangle$ and $|J, 1 - J, 0\rangle$) is split by the interaction energy $2g\sqrt{2J}$. Likewise the triply degenerate level $E = 2\omega$ is split in 3 dressed states with energies $2\omega + 2\sqrt{2J}g, 2\omega, 2\omega - 2\sqrt{2J}g$ as shown above.

The optical spectrum is determined by the allowed transitions between the eigenstates (dressed states): (i) There are two from $|\pm;1\rangle$ to $|-J,0\rangle$ at energies $\hbar(\omega \pm g\sqrt{2J})$. (ii) From $|\pm,0;2\rangle$ to $|\pm;1\rangle$ there are in principle 6 transitions from each of the three states with two excitations to each of the two with one. But for two of them, the transition matrix elements are zero:

$$\langle -;1|\, J_- \,|+;2\rangle = 0 = \langle +;1|\, J_- \,|-;2\rangle$$

for the remaining 4 the transition energies are $\omega \pm g\sqrt{2J}$ (e.g., from $|+;2\rangle$ to $|+;1\rangle$: $(2\omega + 2g\sqrt{2J}) - (\omega + g\sqrt{2J})$ or from $|0;2\rangle$ to $|-;1\rangle$: $2\omega - (\omega - g\sqrt{2J})$). Thus to order $1/N$ only two lines are seen in the emission spectrum, while in the Jaynes-Cummings system (single atom, single cavity)there are already four lines from the three lowest manifolds since the Rabi splitting is $\propto \sqrt{n}$ in the n-excitation manifold.

Further reading: For details on the collective atomic states: Arecchi *et al.* "Atomic Coherent States in Quantum Optics" Phys. Rev. A **6**, 2211, (1972). For a discussion of the spectrum: Yamamoto and Imamoğlu *Mesoscopic Quantum Optics*, ch. 6.4.