Finite propagation velocity of correlations in harmonic bosonic systems

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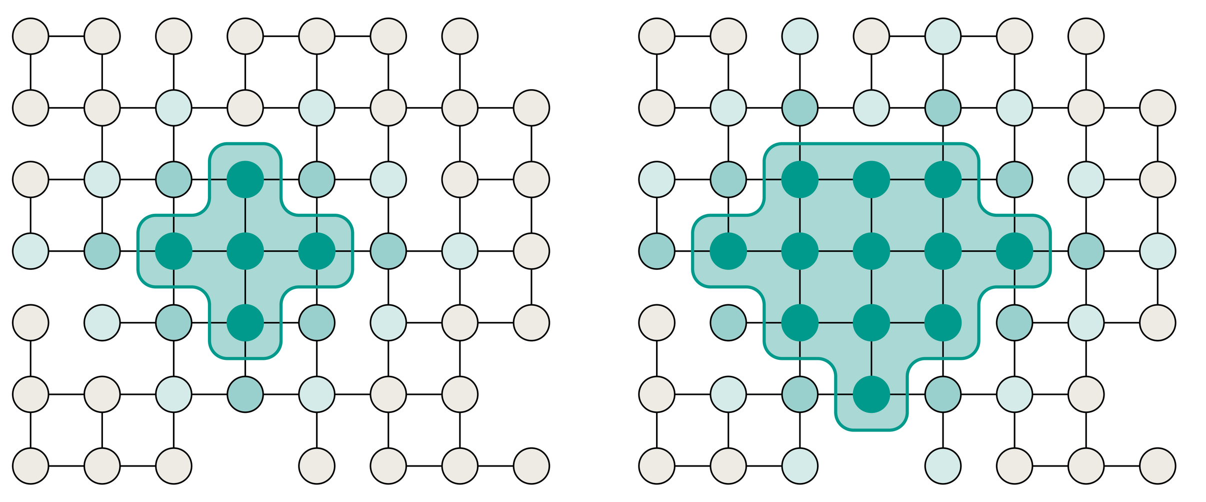
Introduction

For quantum spin systems it has been shown recently that correlations can only propagate through the system with a finite velocity [1]. This proof builds on a result known as **Lieb-Robinson bound** [2, 3]: for a system with short-range interactions the commutator $[\tau_t(A), B]$ of two observables A and B acting on space-like separated spins x and y decays exponentially

$$\|[\tau_t(A), B]\| \le c e^{-\lambda[d(x,y)-v|t|]}.$$

Here $\tau_t(A)$ denotes the usual time evolution in the Heisenberg picture. Thus the Lieb-Robinson bound gives rise to the notion of an **effective causal cone** for a wide class of non-relativistic quantum systems.



Correlations in harmonic bosonic systems spread with a finite maximum velocity. Outside an effective causal cone they decay exponentially.

The above results easily extend to fermionic lattice systems. However, a straightforward generalization to arbitrary bosonic systems is not possible due to the infinite-dimensional Hilbert spaces involved. Interestingly we can still prove a similar upper bound on the propagation velocity of correlations for harmonic bosonic systems.

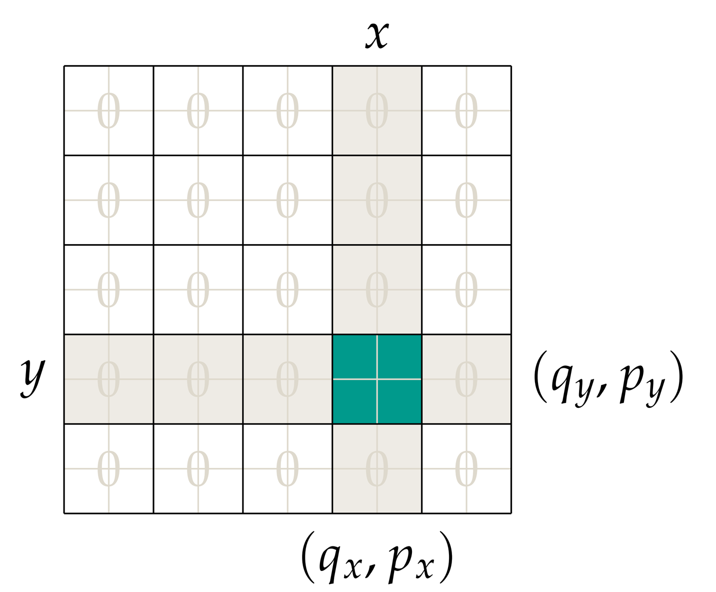
Harmonic bosonic systems

We consider a system of n bosons on a graph $G(V, E)$. The system is described in terms of the canonical operators $R = (q_1, p_1, \ldots, q_n, p_n)$ obeying the commutation relations $[q_k, p_l] = i\delta_{kl}$ where each lattice site corresponds to a bosonic mode (q_k, p_k). Equivalently one may describe the system in terms of annihilation $a_k = 1/\sqrt{2}(q_k + ip_k)$ and creation operators $a_k^\dagger = 1/\sqrt{2}(q_k - ip_k)$.

The simplest class of bosonic systems is characterized by **quadratic Hamiltonians** $H = \sum_{kl} H_{kl} R_k R_l$ with the **Hamiltonian matrix** (H_{kl}). The corresponding time evolution transforms canonical operators like

$$R_k(t) = e^{iHt} R_k e^{-iHt} = \sum_l S_{kl}(t) R_l$$

with $S(t) = e^{\sigma H t}$, the Hamiltonian matrix H and the symplectic form $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of the underlying phase space. The Hamiltonian matrix can be decomposed as $H = \sum_{x,y\in V} h(x,y)$ to reflect the mode structure.



Structure of the matrices $h(x,y)$ and $\Gamma(x,y,t)$ resulting from the mode decomposition of the Hamiltonian and covariance matrix respectively.

Physical realisations include phonons in solids, trapped ions, nanomechanical oscillators and the Bose-Hubbard model $H = -J\sum_{\langle k,l\rangle}(a_k^\dagger a_l + a_k a_l^\dagger) + U/2\sum_k n_k(n_k - 1)$ in the superfluid limit where $U = 0$.

Covariance matrix and correlations

In the phase space description every state ρ of a bosonic system can be characterized in terms of its moments. First and second moments correspond to simple expectation values of the canonical operators and are called **displacement vector** $d_k := \mathrm{tr}[\rho R_k] = \langle R_k\rangle$ and **covariance matrix**

$$\gamma_{kl} := \langle\{R_k - d_k, R_l - d_l\}\rangle$$

respectively. Thus connected correlation functions of second order are directly given by the elements of the covariance matrix. Similar to the Hamiltonian the covariance matrix can also be split into its mode constituents $\gamma(t) = \sum_{x,y\in V}\Gamma(x,y,t)$. The maximal correlations between any two modes separated by the distance d is then given by $C_d(t) := \max_{d(x,y)=d}\|\Gamma(x,y,t)\|$.

Finite propagation velocity

The interaction strength is measured by the following norm:

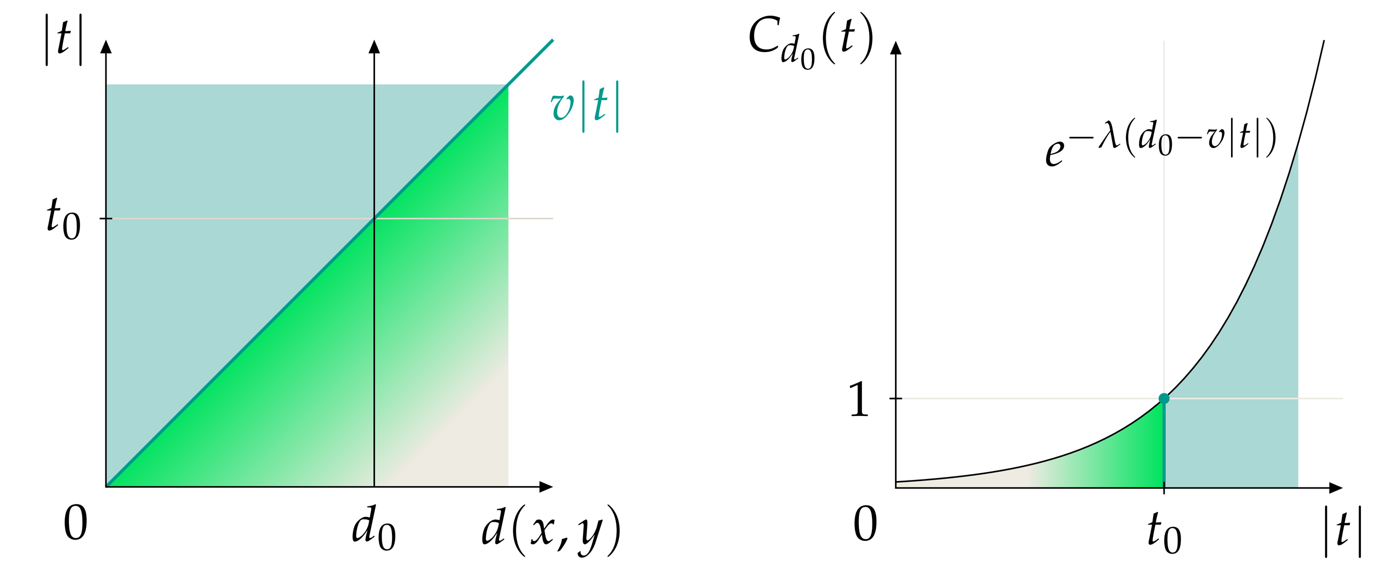
$$\|h\|_\lambda := \max_l \sum_{k=0}^{d_G} e^{\lambda|l-k|} \max_{x\in V} \sum_{m=|l-k|}^{l+k} \sum_{\substack{y\in V \\ d(x,y)=m}} \|h(x,y)\|.$$

If $\lim_{d_G\to\infty}\|h\|_\lambda < \infty$ and the initial state ρ is a product state connected correlation functions of second order between space-like separated modes x and y decay exponentially

$$\|\Gamma(x,y,t)\| \le C\, e^{-\lambda[d(x,y)-v|t|]}$$

where $C > 0$ is a trivial factor and the velocity is given by $v = \inf_\lambda \|h\|_\lambda/\lambda$.

This result easily generalizes to initial states with essentially finite correlation length. Note that the velocity v scales linearly with the Hamiltonian. For **Gaussian states** which are solely characterized by their first and second moments this result actually applies to higher order correlations as well since these can be expressed completely in terms of the covariance matrix.



Effective causal cone for correlations is shown on the left. For fixed $d(x,y) = d_0$ the exponential bound on the maximal correlations $C_{d_0}(t)$ is displayed on the right (with $C = 1$ for simplicity).

Simple example

Consider the case of a one-dimensional chain with next-neighbour interactions only. This means that the individual terms $\|h(x,y)\|$ of the Hamiltonian can be bounded from above by some constant h if $d(x,y) \le 1$. In all other cases $\|h(x,y)\| = 0$ holds. One can then easily show that the interaction strength is bounded by

$$\|h\|_\lambda \le 3h(1 + 2e^\lambda)$$

which is independent of the system size. This leads to a simple upper bound on the propagation velocity of correlations $v \le 20h$.

References

[1] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).

[2] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. **28**, 251 (1972).

[3] B. Nachtergaele and R. Sims, Commun. Math. Phys. **265**, 119 (2005).