1. *(equivalence of coherent states and "classical fields + vacuum")* Let H denote the minimal coupling Hamiltonian for a system of charges in the quantized field $\mathbf{A}$ and a (classical) external field $\mathbf{A}_{\text{ext}}$:

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left[\mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{r}_\alpha) + \mathbf{A}_{\text{ext}}(\mathbf{r}_\alpha)\right]^2 + \sum_{\alpha\neq\beta} \frac{q_\alpha q_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|} + \hbar \sum_j \omega_j (a_j^\dagger a_j + \frac{1}{2}).$$

The aim of this exercise if to prove the "equivalence of coherent states and classical fields" in the following sense: Show that the Hamiltonian $H_0 = H(\mathbf{A}_{\text{ext}} = 0)$ (i.e. in the absence of an external field) applied to a coherent state of the radiation field (and an arbitrary state of the charges) generates the same dynamics as $H_1 = H(\mathbf{A}_{\text{ext}})$ (i.e. with a non-zero external field), when applied to a radiation field in the vacuum.

Let in the following

$$|\{\alpha_j\}\rangle = \prod_j |\alpha_j\rangle$$

denote a multi-mode coherent state (the index j runs through the discrete set of modes with eigenfrequency ω_j). The main idea of the proof is to relate the two initial situations by a time-dependent unitary transformation $T(t)$. The proof proceeds in four steps:

a) Consider the displacement operator

$$T(t) = \prod_j \exp[\alpha_j^* e^{i\omega_j t} a_j - \alpha_j e^{-i\omega_j t} a_j^\dagger]$$

Show that it transforms the field operator $\mathbf{A}(\mathbf{r}_\alpha)$ as

$$T(t)\mathbf{A}(\mathbf{r})T(t)^\dagger =)\mathbf{A}(\mathbf{r}) + \mathbf{A}_{\text{ext}}(\mathbf{r}),$$

with a function $\mathbf{A}_{\text{ext}}(\mathbf{r})$ depending on the α_j but not on the operators $a_j, a_j^\dagger$.

b) $T(t)$ is a unitary operation, hence it can be seen as a change of basis. How does the initial state $|\{\alpha_j\}\rangle$ look when it is expressed in the new basis?

c) Prove that in the new basis, the dynamics is given by

$$i\hbar \frac{d}{dt}|\phi\rangle = \left(T(t)H_0T(t)^\dagger + i\hbar\left[\frac{dT(t)}{dt}T(t)^\dagger\right]\right)|\phi\rangle$$

d) With the result from (a) and the transformations properties of the a_j from exercises # 03, calculate $H(t) = T(t)H_0T(t)^\dagger$ and $H(t) + \frac{dT(t)}{dt}T(t)^\dagger$. (The BCH formula may be useful again.) Interpret the result.

2. *(vacuum field correlation function)* Consider the electric field operator (in the finite-volume quantization) in the Heisenberg picture:

$$E(t, \mathbf{r}) = i \sum_j \mathcal{E}_j \left[a_j(0) \boldsymbol{\epsilon}_j e^{i(\mathbf{k}_j \cdot \mathbf{r} - \omega_j t)} - \text{h.c.} \right],$$

where $\mathcal{E}_j = \left(\frac{2\pi\hbar\omega_j}{V}\right)^{1/2}$ and V is the quantization volume.

compute the vacuum expectation values

$$\langle 0 | E_m(t_1, \mathbf{r}_1) E_n(t_2, \mathbf{r}_2) + E_n(t_2, \mathbf{r}_2) E_m(t_1, \mathbf{r}_1) | 0 \rangle,$$

where $n, m = x, y, z$ refers to the cartesian components of the field. (Recall that $\sum_j$ stands for $\sum_{\mathbf{k}_j} \sum_{\boldsymbol{\epsilon}_1(\mathbf{k}_j), \boldsymbol{\epsilon}_2(\mathbf{k}_j)}$) and use that

$$\sum_{\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_1(\mathbf{k}), \boldsymbol{\epsilon}_2(\mathbf{k})} \epsilon_m \epsilon_n = \delta_{nm} - \kappa_m \kappa_n,$$

where $\boldsymbol{\kappa}$ is the unit vector in $\mathbf{k}$-direction (which is a consequence of the transversality of $\mathbf{E}$). This allows to perform the sum over $\boldsymbol{\epsilon}$.

Replacing summation with integration in the usual way and denoting $\tau = t_1 - t_2$ and $\boldsymbol{\rho} = \mathbf{r}_1 - \mathbf{r}_2$

$$\sum_{\mathbf{k}} \left(\frac{2\pi}{L}\right)^3 \leftrightarrow \int d^3k$$

one obtains

$$\langle 0 | E_m(t_1, \mathbf{r}_1) E_n(t_2, \mathbf{r}_2) + E_n(t_2, \mathbf{r}_2) E_m(t_1, \mathbf{r}_1) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} (\omega^2 \delta_{mn} - c^2 k_m k_n) \frac{e^{i(\mathbf{k} \cdot \boldsymbol{\rho} - \omega\tau)}}{\omega} + \text{c.c.}$$

The integral can be understood as a derivative of the function

$$D_+(\boldsymbol{\rho}, \tau) = ic \int \frac{d^3k}{(2\pi)^3} \frac{e^{i(\mathbf{k} \cdot \boldsymbol{\rho} - \omega\tau)}}{\omega}$$

namely with $iD_1 = \text{Im} D_+$

$$-\frac{\hbar}{\epsilon_0 c} \left(\delta_{nm} \frac{\partial^2}{\partial \tau^2} - c^2 \frac{\partial}{\partial \rho_m} \frac{\partial}{\partial \rho_n} \right) D_1(\boldsymbol{\rho}, \tau).$$

D_1 can be computed (in spherical coordinates and by introducing a convergence factor $\lim_{\eta \to 0^+} e^{-\eta k}$) to be

$$\begin{aligned} D_1 &= \lim_{\eta \to 0^+} \frac{1}{(2\pi)^2 \rho} \left[\frac{\rho - c\tau}{\eta^2 + (\rho - c\tau)^2} + \frac{\rho + c\tau}{\eta^2 + (\rho + c\tau)^2} \right] \\ &= \lim_{\eta \to 0^+} \frac{1}{2\pi^2} \frac{\rho^2 - c^2\tau^2 + \eta^2}{(\rho^2 - c^2\tau^2)^2 + 2\eta^2(\rho^2 + c^2\tau^2) + \eta^4} \end{aligned}$$

Off the light cone ($\rho^2 \neq c^2\tau^2$) we can take the limit $\eta \to 0^+$ and the resulting derivative is easy to calculate. The $\boldsymbol{\rho} = 0$ correlation function (which is the correlation function of the "reservoir" that an atom at $\mathbf{r} = 0$ sees). Show that for large τ the autocorrelation function ($n = m$) decays as τ^{-4} and that the decay is more rapid for $n \neq m$. Plot the result for small but finite η to see the δ-like behavior of the correlation function.

Similar results hold for **B** and the correlations between **E** and **B**.

3. *(Master equation for two-level atom)* The reduced dynamics of a two-level atom (with ground state $|g\rangle$ and excited state $|e\rangle$) in the vacuum field is described by the Master equation

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}[H_0, \rho] - \frac{\Gamma}{2}\left[\sigma_{ee}\rho + \rho\sigma_{ee} - 2\sigma_{ge}\rho\sigma_{eg}\right],$$

where $H_0 = \hbar\omega/2(|e\rangle\langle e| - |g\rangle\langle g|)$ and $\sigma_{ee} = |e\rangle\langle e|$ and $\sigma_{ge} = \sigma_{eg}^\dagger = |g\rangle\langle e|$. Find the populations $\rho_{ee}(t), \rho_{gg}(t)$ and coherences $\rho_{eg}(t)$.