

1. (*Hadamard Test*) Suppose we have an observable U with eigenvalues ± 1 , so that U is both hermitian and unitary. Show that the following circuit (where the measurement in the first qubit is a Z measurement) realizes an U -measurement on the qubit $|q_1\rangle$. What if $|q_1\rangle$ corresponds to the state of n qubits?

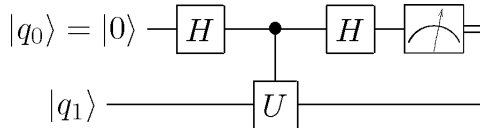


Figure 1: Hadamard test

Variations of this setup can be found in many quantum algorithms. It can both be used to measure properties of $|q_1\rangle$ (if U is known) or of U (if $|q_1\rangle$ is known). We consider two examples.

2. (*Fidelity, Purity*) Let the initial state in the second system be not a pure state $|q_1\rangle$ but two arbitrary states ψ_1, ψ_2 , i.e. $|q_1\rangle \rightarrow |\psi_1\rangle |\psi_2\rangle \in \mathcal{H} \otimes \mathcal{H}$ and let U be the SWAP gate on $\mathcal{H} \otimes \mathcal{H}$, i.e.

$$U \equiv U_{\text{SWAP}} : |k\rangle |l\rangle \mapsto |l\rangle |k\rangle \forall k, l$$

for some orthonormal basis (ONB) $|k\rangle$ of \mathcal{H} .

(i) Compute the expectation value of the observable Z in the control bit ($\langle Z \rangle_{\text{out}} = \text{probability to find "1"} - \text{probability to find "0"}$).

(ii) Show that the result is 1 if and only if $|\psi_1\rangle = |\psi_2\rangle$. This implements a measurement of the *transition probability* between the two states [which is often called the *fidelity* of $|\psi_1\rangle$ with respect to $|\psi_2\rangle$ (or vice versa) and is a frequently used distance measure in QIT].

(iii) Use the above result to compute $\langle Z \rangle_{\text{out}}$ for an initial state in which $|q_1\rangle$ is replaced by two copies of some mixed state ρ : $|q_1\rangle \rightarrow \rho \otimes \rho$. Show that the result is 1 if and only if ρ is pure. This quantity is called the *purity* of the state ρ .

3. (*Quantum Phase Estimation*): As we have seen, the state $|+\rangle \sim |0\rangle + |1\rangle$ is unchanged by the controlled- U operation if the state on which U act is an $+1$ -eigenstate of U and it is flipped to $|-\rangle$ if $U|q_1\rangle = -|q_1\rangle$. Now consider $U = \exp(i2\pi\theta Z)$, i.e. having eigenvectors $|0\rangle, |1\rangle$ with eigenvalues $u_j = e^{\pm i2\pi\theta_j}, j = 0, 1$.

We want to determine θ_0 , the phase of the eigenvalue u_0 . Let's assume that we know θ_0 to lie in the set $\{0, 1/4, 1/2, 3/4\}$. Show that the above circuit with $U \rightarrow U^2$ allows to distinguish between the subsets $\{0, 1/2\}$ and $\{1/4, 3/4\}$. *Hint: see figure 2*

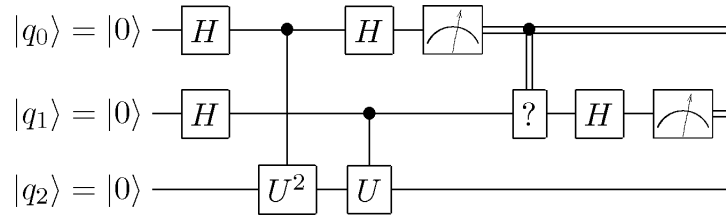


Figure 2: Estimating U eigenvalue

Sketch how this procedure could be extended to measure arbitrary eigenvalues $\theta_0 \in \{l/2^n\}$ for integer $l < 2^n$. How does the number M of control qubits needed scale with the exponent n ? Remark: this algorithm can be applied even if θ_0 is not a rational number as above. In this case, one needs about twice as many qubits to reach the same precision.