1. **Optical Bloch equations for a two-level atom**
   Consider a two-level atom coupled to the free quantum field and driven by an external field governed by the Hamiltonian

$$H = H_A + H_R - \mathbf{d} \cdot [\mathbf{E}_{\text{ext}}(0,t) + \mathbf{E}_{\perp}(0,t)]$$

   ($H_A = \hbar\omega_0\sigma_z$ internal atomic part, H_R free radiation field) with $\mathbf{E}_{\text{ext}}(0,t) = \mathbf{E}_0 \cos(\omega_L t)$. This combines the processes discussed in the exercises #3, 3 and #4, 3.

   In the "approximation of independent rates of variation" we can express the time derivatives of the elements of the density matrix

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$$

   by adding the terms obtained if either of the two couplings $\mathbf{d}\cdot\mathbf{E}_{\text{ext(0,t)}}$ or $\mathbf{d}\cdot\mathbf{E}_{\perp}(0,t)$ acted alone.

   a) Why is this an approximation? In what limit of the occuring parameters (Rabi frequency Ω, transition frequency ω_0, laser frequency ω_L) do you expect this to be a good approximation?
   b) perform the rotating wave approximation and make the coefficients of the resulting differential equation time-independent by transforming to a rotating frame $\tilde{\rho}_{ab} = \rho_{ab} e^{i\omega_L t} = (\tilde{\rho}_{ba})^*$. Rewrite them for the "Bloch vector"

$$\mathbf{u} := \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \tilde{\rho}_{ab} + \tilde{\rho}_{ba} \\ (\tilde{\rho}_{ab} - \tilde{\rho}_{ba})/i \\ \tilde{\rho}_{bb} - \tilde{\rho}_{aa} \end{pmatrix},$$

   i.e., the expectation value of the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ (in the rotating frame), which fully characterizes the state.
   c) Discuss the time-evolution in the cases (i) $\omega_0 = \omega_L, \Omega \ll \Gamma$, (ii) $\omega_0 = \omega_L, \Omega \gg \Gamma$, and (iii) $|\omega_0 - \omega_L| \gg \Omega, \Gamma$.
   d) Find the steady state solution $\frac{d}{dt}\mathbf{u} = 0$.

2. **C-NOT gate without auxiliary levels**
   The aim of this exercise is to derive a simpler, more robust realization of the C-NOT gate which does not need auxiliary levels.

   a) Assume that a trapped ion can be addressed individually and a C-NOT between the ion and a normal mode of the ion chain can be realized, i.e., the transformation

$$\begin{aligned} |0\rangle\,|\downarrow\rangle_j &\rightarrow |0\rangle\,|\downarrow\rangle_j \\ |0\rangle\,|\uparrow\rangle_j &\rightarrow |0\rangle\,|\uparrow\rangle_j \\ |1\rangle\,|\downarrow\rangle_j &\rightarrow |1\rangle\,|\uparrow\rangle_j \\ |1\rangle\,|\uparrow\rangle_j &\rightarrow |1\rangle\,|\downarrow\rangle_j \end{aligned}$$

Show that this gate, combined with two SWAP-gates between ion j and the oscillator mode allows to build an C-NOT between ions i and j.

b) To realize the above gate, consider a *resonant* ("carrier") *traveling* wave $\mathbf{E}(x,t) = \mathbf{E}_0 \cos(kx - \omega_L t + \Phi)$ interacting with the jth ion in the dipole and rotating wave approximation (for terms oscillating at ω_L). Do *not* make the Lamb-Dicke approximation to keep only terms linear in x. Show that the states $|n\rangle\,|\downarrow\rangle$ and $|n-1\rangle\,|\uparrow\rangle$ are coupled with a Rabi frequency $\Omega_{n,n} = |\,\langle n|\,H_{\text{int}}^{(j)}\,|n\rangle\,| = g_j|\,\langle n|\,e^{i\eta(a+a^\dagger)}\,|n\rangle\,|$, where η is the Lamb-Dicke parameter and g_j the resonant Rabi frequency of the transition $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$.

c) For harmonic oscillator modes $|n\rangle$ this matrix element can be calculated as a function of η as $\Omega_{0,0} = g_j e^{-\eta^2/2}$ and in general $\Omega_{n,n} = g_j e^{-\eta^2/2} L_n(\eta^2)$, where L_n is the nth Laguerre polynomial. Show that by choosing η such that $\Omega_{1,1}/\Omega_{0,0} = (2l+1)/(2m)$ for integers $l > m \geq 0$ and driving the carrier transition for a time $\tau = (l+1/2)\pi/\Omega_{1,1}$ one obtains the desired C-NOT gate between mode and ion (up to local phase factors).