The retarded effects of an electron bunch

K. Iqbal and H. Ruhl
Computational & Plasma Physics, LMU, München, Germany

1. Introduction and motivation

In both laser wakefield and laser acceleration in vacuum the accelerating electrons oscillate due to transverse electric field components (in laser wakefield only off-axis electrons exhibit betatron oscillations) [1,2]. The oscillating electrons emit radiation and these radiation react back on them. For Low energies electron beam, the radiation effects are small, however, for high energies, the effects of emitted radiation become severe. For example, in a Laser Wakefield Accelerator (LWFA) the transverse focusing field is of the order of the longitudinal field. The trapped electrons exhibit betatron oscillations due to the strong focusing force along with the accelerating force. The oscillating electrons radiate in a similar fashion as do electrons in an undulator field [1]. The amount of energy radiated by an electron can be substantial. It may have a significant effect on the evolution of the electron beam.

Currently relativistic electron bunches with charge >pC are being produced in the laser wakefield [10,11,12], which means that it contains $> 10^9$ particles. One feels that if the radiation of a single electron inside the beam affects its own motion [2,4,5,6,13] then it should affect the motion of the other electrons of the beam for certain given parameters. Therefore, there is a need to investigate the accumulated effects of radiation on electron motion.

2. Equation of motion of an electron

The Lorentz-Abraham-Dirac (LAD) equation describes the motion of a radiating electron

$$m\dot{u}^{\alpha} = -\frac{e}{c}F^{\alpha\beta}u_{\beta} + m\tau_o\left(\ddot{u}^{\alpha} - \dot{u}^{\beta}\dot{u}_{\beta}u^{\alpha}/c^2\right), \quad (1)$$

where $\tau_o = e^2/6\pi\epsilon_o mc^3$. However, It has unphysical solutions like: preacceleration and runaway (for review see [8]). Later on some other models have been developed to avoid these unphysical solutions by Landau and Lifshitz [9], Mo and Papas , Caldirola, Yaghjian and Sokolov [8]. However, the Landau-Lifshitz (LL) equation provides a good first order approximation of the LAD equation. It can be obtained by the perturbative expansion of the equation of motion [9]

$$m\dot{u}^{\alpha} = -\frac{e}{c}F^{\alpha\beta}u_{\beta} - \frac{e}{c}\tau_o\left[\frac{\partial F^{\alpha\beta}}{\partial x^{\gamma}}u^{\gamma}u_{\beta} - \frac{e}{mc}\left(F^{\alpha\beta}F_{\beta\gamma}u^{\gamma} - F^{\beta\gamma}F_{\gamma l}u^{l}u_{\beta}u^{\alpha}\right)\right], \quad (2)$$

it does not have unphysical solutions. The above perturbative expansion of the LAD equation is valid if $F_{rad} \ll F_L$. The radiation fields produced by a radiating electron are known as Lienard-Wiechert fields and are given by

$$\vec{E}_{LW} = \frac{e}{4\pi\epsilon_o}\left(\frac{\hat{n} - \vec{\beta}(t')}{\gamma^2(1 - \hat{n}\cdot\vec{\beta}(t'))^3R^2(t')} + \frac{1}{c}\frac{\hat{n}\times(\hat{n} - \vec{\beta}(t'))\times\dot{\vec{\beta}}(t')}{(1 - \hat{n}\cdot\vec{\beta}(t'))^3R(t')}\right), \quad (3)$$

$$\vec{B}_{LW} = \vec{k}\times\vec{E}_{LW}, \quad (4)$$

The retarded quantities $(\vec{R}(t'), \vec{\beta}(t'), etc)$ can be expanded about the current time t for electron bunches of much smaller duration

$$R(t') = R - \dot{R}\frac{R}{c} + \frac{\ddot{R}}{2}\left(\frac{R}{c}\right)^2 - \frac{\dddot{R}}{6}\left(\frac{R}{c}\right)^3 + \cdots + \text{nonlinear terms.} \quad (5)$$

the $R(t)$ is given by the relation [7]

$$R = R\beta + d, \quad \Rightarrow R = \frac{d}{1 - \beta_z}, \quad R^2 = (R\beta)^2 + d^2, \quad \Rightarrow R = \frac{d}{\sqrt{1 - \beta_z^2}} \quad (6)$$

3. Analytical solution

The Landau-Lifshitz equation for a particle under radiation effects of other particles in three vector notation can be written as

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \vec{E}_{rad}^{(other)} + \vec{v}\times(\vec{B} + \vec{B}_{rad}^{(other)})\right) + \tau_o e\gamma\frac{d}{dt}(\vec{E} + \vec{v}\times\vec{B}) + \frac{\tau_o e^2}{mc}\Big[c(\vec{E}\times\vec{B} + (\vec{v}\times B)\times\vec{B}) + (\vec{v}/c\cdot\vec{E})\vec{E} - \frac{\vec{v}}{c}\gamma^2\left((\vec{E} + \vec{v}\times\vec{B})^2 - (\vec{v}/c\cdot\vec{E})^2\right)\Big]. \quad (7)$$

The subscript "*rad*" in equation (7) stands for retarded fields. For linearly polarized plane wave equation (7) can be written as

$$\frac{dp_x^{(2)}}{d\tau} = -a_o\cos(\tau) + a_o\tau_o K\sin(\tau) - \frac{\sigma}{K}\frac{dp_x^{(1)}}{d\tau} - a_o^2\tau_o p_x K\cos^2(\tau), \quad (8)$$

$$\frac{dp_x^{(1)}}{d\tau} = -a_o\cos(\tau) + a_o\tau_o K\sin(\tau) - a_o^2\tau_o p_x K\cos^2(\tau), \quad (9)$$

$$\frac{dK}{d\tau} = -a_o^2\tau_o K^2\cos^2(\tau), \quad (10)$$

where $a_o = eE_o/(m_e\omega_L c)$ is the normalized laser intensity, $\sigma = Ze^2/(4\pi\epsilon_o m_e c^2 d)$, $\gamma - p_z = K$, and $\tau = t - z$. We have also assumed that for an electron moving in the $-z$ direction with large initial energy $v_x \ll v_z$. The solutions of equations (8)-(10) give the value of $p_x^{(2)}$

$$p_x^{(2)} = \frac{1}{180(1 + 2\tau\eta + \eta\sin(2\tau))}\Big(60(4a_o\eta(1 + \Lambda(3 - 2\tau\eta)) + 3(1 + \Lambda - 2\tau\eta\Lambda)\zeta) - 90(a_o\eta(3 + 8\Lambda + 16\tau\eta\Lambda) + 2(1 + \Lambda + 2\tau\eta\Lambda)\zeta)\cos(\tau) + 30a_o\eta\cos(3\tau) - 30(a_o(6(1 + \Lambda) + \eta(12\tau + 24\tau\Lambda - 71\eta\Lambda + 24\tau^2\eta\Lambda)) - 33\eta\Lambda\zeta)\sin(\tau) - 60\eta\Lambda(4a_o\eta + 3\zeta)\sin(2\tau) + 5\eta\Lambda(17a_o\eta + 6\zeta)\sin(3\tau) + 3a_o\eta^2\Lambda\sin(5\tau)\Big), \quad (11)$$

$$\gamma = \frac{1 + p_x^2}{2K} + \frac{K}{2} \quad \text{and ,} \quad p_z = \frac{1 + p_x^2}{2K} - \frac{K}{2} \quad (12)$$

And for circularly polarized plane wave

$$\frac{dp_x^{(2)}}{d\tau} = -a_o\sin(\tau) - a_o\tau_o K(\cos(\tau) + a_o p_x^{(2)}) - \frac{\sigma}{K}\frac{dp_x^{(1)}}{d\tau}, \quad (13)$$

$$\frac{dp_y^{(2)}}{d\tau} = -a_o\cos(\tau) + a_o\tau_o K(\sin(\tau) - a_o p_y^{(2)}) - \frac{\sigma}{K}\frac{dp_y^{(1)}}{d\tau}, \quad (14)$$

$$\frac{dp_x^{(1)}}{d\tau} = -a_o\sin(\tau) - a_o\tau_o K\cos(\tau) - a_o^2\tau_o K p_x^{(1)}, \quad (15)$$

$$\frac{dp_y^{(1)}}{d\tau} = -a_o\cos(\tau) + a_o\tau_o K\sin(\tau) - a_o^2\tau_o K p_y^{(1)}, \quad (16)$$

$$\frac{dK}{d\tau} = -a_o^2\tau_o K^2, \quad (17)$$

and solution is given by

$$p_x^{(2)}(\tau) = \frac{1}{1 + \eta_o\tau}\Big(a_o(-1 + (-1 + \eta_o(\tau + 4\eta_o))\Lambda) + 2\eta_o\Lambda\zeta + (a_o(1 + \Lambda + \eta_o(\tau + 2\Lambda\tau - 4\eta_o\Lambda + \eta_o\Lambda\tau^2)) - 2\eta_o\Lambda\zeta)\cos(\tau) - (a_o\eta_o(1 + 3\Lambda + 3\eta_o\Lambda\tau) + (1 + \Lambda + \eta_o\Lambda)\zeta)\sin(\tau))\Big), \quad (18)$$

$$p_y^{(2)}(\tau) = \frac{1}{(1 + \eta_o\tau)}(a_o\eta_o(1 + (3 - \tau\eta_o)\Lambda) + (1 + \Lambda - \tau\eta_o\Lambda)\zeta - (a_o\eta_o(1 + 3\Lambda + 3\tau\eta_o\Lambda) + (1 + \Lambda + \tau\eta_o\Lambda)\zeta)\cos(\tau) - (a_o(1 + \Lambda + \eta_o(\tau + 2\tau\Lambda - 4\eta_o\Lambda + \tau^2\eta_o\Lambda)) - 2\eta_o\Lambda\zeta)\sin(\tau)), \quad (19)$$

$$\gamma = \frac{1 + p_x^2 + p_y^2}{2K} + \frac{K}{2} \quad \text{and ,} \quad p_z = \frac{1 + p_x^2 + p_y^2}{2K} - \frac{K}{2}, \quad (20)$$

equation (20) gives the longitudinal momentum and energy of the particle P_2. We have studied the motion of an electron counter propagating to the laser pulse as shown in Figure 1.

4. Results

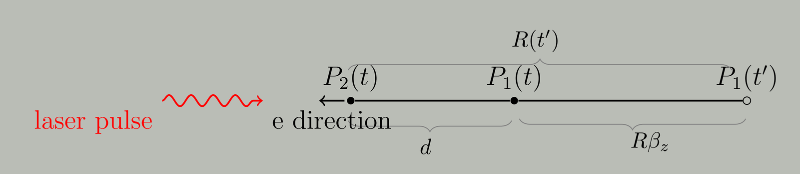
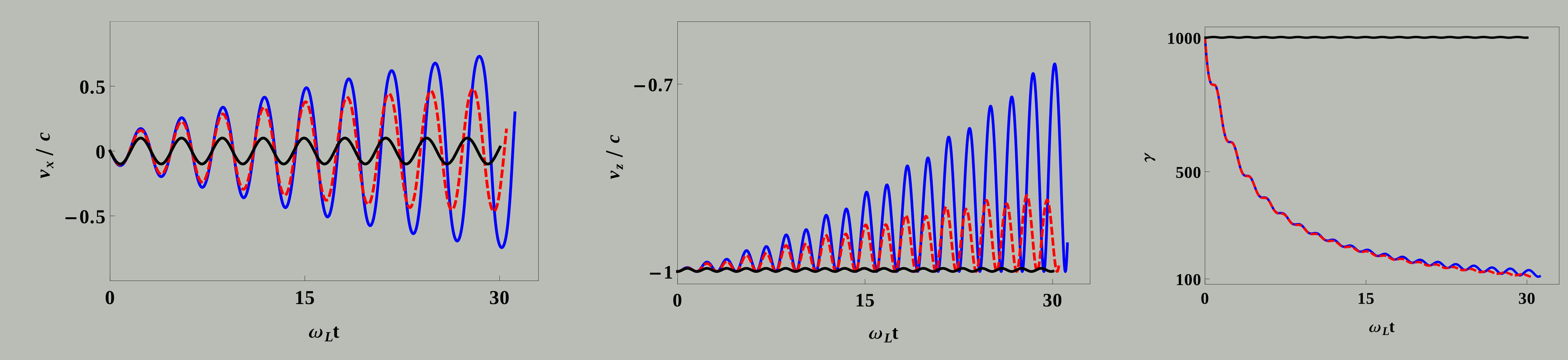
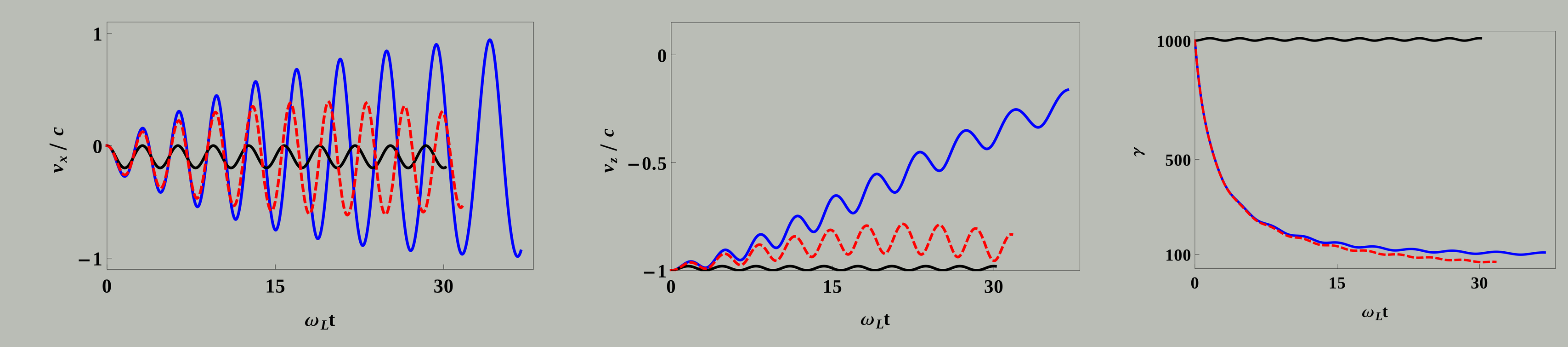


Figure 1: The electron laser interaction model



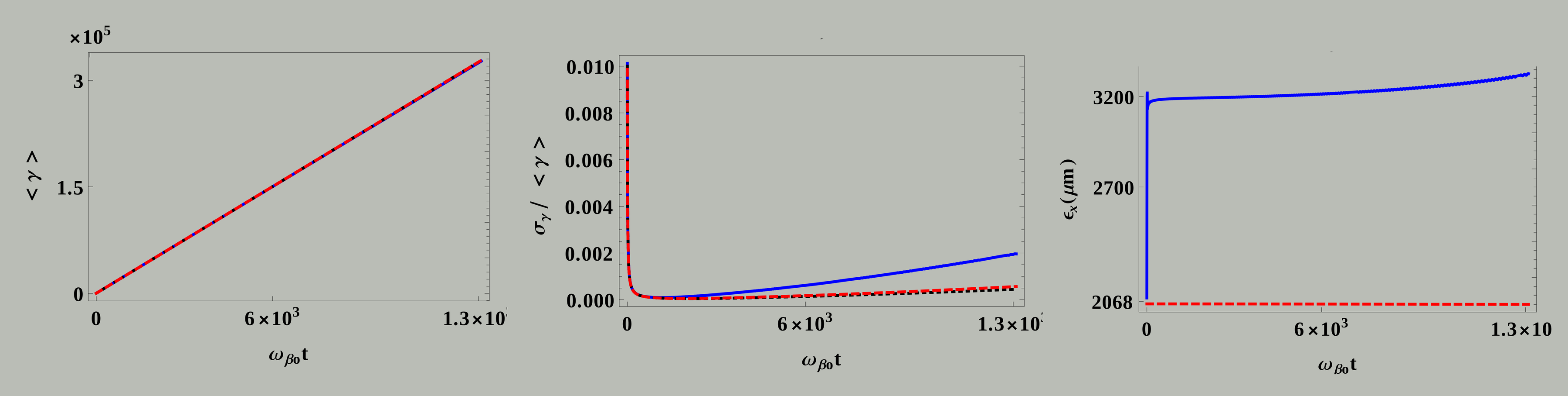
(a) (b) (c)

Figure 2: (a) The transverse velocity v_x, (b) the longitudinal velocity v_z, and (c) the Lorentz factor γ as a function of lab time t of particle P_2 with $\gamma_o = 1000$ for a linearly polarized wave with $a_o = 100$. The red dashed line represents the motion with the LL equation plus retarded effects, solid blue line stands for the LL equation and solid black line shows motion by the Lorentz equation.



(a) (b) (c)

Figure 3: (a) The transverse velocity v_x, (b) the longitudinal velocity v_z, and (c) the Lorentz factor γ as a function of lab time t of particle P_2 with $\gamma_o = 1000$ for a circularly polarized wave with $a_o = 100$. The colour scheme as mention earlier



(a) (b) (c)

Figure 4: The electron beam dynamics. (a) the mean energy $\langle\gamma\rangle$, (b) the relative energy spread $\sigma_\gamma/\langle\gamma\rangle$, and (c) the normalized transverse emittance ϵ_x versus $\omega_{\beta 0}t$ of the beam. The electron beam moves with initial energy $\langle\gamma_o\rangle = 200$, initial emittance $\varepsilon_o = 2068\mu m$. Initial bunch charge is 1.6 nC. The solid blue and red-dashed lines stand for the corresponding quantities with and without out retarded effects respectively.

Electron bunches with the charge of few pC with bunch length of less than μm are being produced in the laser wakefields [11,12]. The radiation effects of such bunches on the subsequent motion of the electrons are studied in the laser pulses of different polarization and also in the laser wakefields. It is found that the retarded effects metigate the self force effects for the electron counter propagating to the laser pulse. The retarded effects are represented by red-dashed line in Figures 2, 3.

In the case of laser wakefield the retarded fields reduce the energy gain and increase the relative energy spread and transverse emittance of the beam of relatively high initial radius and high total bunch charge, (blue line in Figure 4, a, b, c). However, if the electron beam has low charge or small initial radius, the retarded effects are negligible.

5. Summary

- The retarded effects metigate the self force effects for both kind of polarization.
- The retarded fields reduce the energy gain of the beam and increase the energy spread and transverse emittance of the beam in a laser wakefield accelerator.
- The retarded effects depend linearly on the total charge and the length of the bunch.

6. References

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